Math 1A: Quiz 8

ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) Use calculus to carefully sketch a graph of the following function:

$$y = (x^2 - 3x + 2)e^x$$

Label your graph with the x values of all zeros, local extrema, and inflection points. Also label the y-intercept and any horizontal asymptotes. Use the back of this page to show your (well-organized and clearly labeled) work.

(On the original quiz, the grid lines took up the entire page. We have shrunk them for this answer key.)

We begin by finding the 0s of $y(x)$. Since e^x is never 0, these correspond to roots of $(x^2 - 3x + 2) = (x-2)(x-1)$, so the zeros are at $x = 1$ and $x = 2$. Moreover,

$$\lim_{x \to -\infty} (x^2 - 3x + 2)e^x = 0,$$

so there is a horizontal asymptote to the left, but

$$\lim_{x \to +\infty} (x^2 - 3x + 2)e^x = \infty,$$

so there is none to the right.

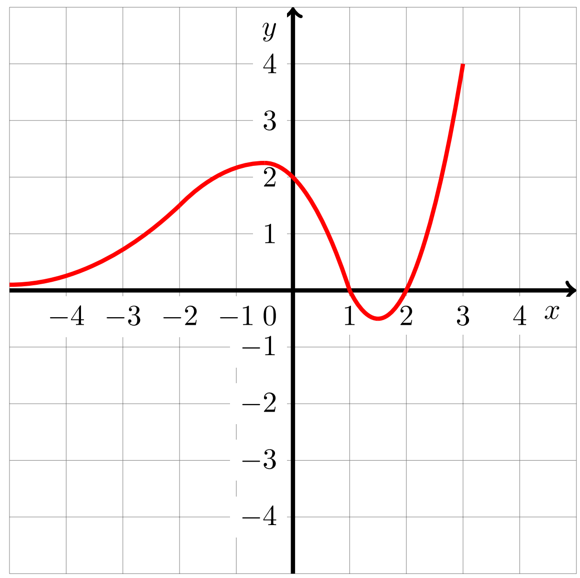
The two zeros and the horizontal asymptote imply that there should be at least two critical points, using Rolle's Theorem on $(-\infty, 1]$ and on $[1, 2]$, although we have not formally justified the use of Rolle's Theorem when one endpoint is $\pm\infty$.

We now differentiate $y = (x^2 - 3x + 2)e^x$ to find the locations of the critical points.

$$\begin{aligned} y' &= (2x-3)e^x + (x^2 - 3x + 2)e^x \\ &= (x^2 - x - 1)e^x \end{aligned}$$

We want to solve $y' = 0$. The quadratic $(x^2 - x - 1)$ does not factor in integers, but the quadratic formula gives:

$$x = \frac{1 \pm \sqrt{5}}{2}$$



Lastly, we solve $y'' = 0$:

$$\begin{aligned} y'' &= (2x-1)e^x + (x^2 - x - 1)e^x \\ &= (x^2 + x - 2)e^x \\ 0 &= x^2 + x - 2 \\ &= (x+2)(x-1) \\ x &= -2 \text{ or } 1 \end{aligned}$$

We can either check signs of the various derivatives of y, or use the fact that $y > 0$ if $x > 2$ or if $x < 1$ and $y < 0$ if $1 < x < 2$ to guess that $x = (1 - \sqrt{5})/2 \approx -0.5$ is a local maximum, and $x = (1 + \sqrt{5})/2 \approx 1.5$ is a local minimum, where we have used $\sqrt{5} \approx 2$. The critical points are at -2 and 1. The y-intercept is at $(0 - 0 + 2)e^0 = 2$. Thus we have the above graph.

2. (bonus) What are your Spring Break plans?