Math 1A: Quiz 9 — ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) Evaluate the following limit:

$$\lim_{x\to\infty} (\sinh x)^{1/x}$$

Hint: For this problem, you only need to apply L'Hospital's rule once.

For any limit, we always begin by doing the most naive thing: we try to evaluate it directly. $\sinh\infty = \infty$, and $1/\infty = 0$, so this limit is behaving like ∞^0, an indeterminate form. Thus, we must manipulate it and then possibly apply L'Hospital's rule.

We begin manipulating by rewriting the exponential in terms of es and logarithms:

$$\lim_{x\to\infty} (\sinh x)^{1/x} = \lim_{x\to\infty} e^{\ln\left((\sinh x)^{1/x}\right)} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(\sinh x)} = e^{\lim\limits_{x\to\infty}\left[\frac{1}{x}\ln(\sinh x)\right]}$$

Thus we have a NEW PROBLEM:

$$\lim_{x\to\infty}\left[\frac{1}{x}\ln(\sinh x)\right]$$

When we try to plug in "$x = \infty$", we see that $\frac{1}{\infty} = 0$, and $\ln\sinh\infty = \ln\infty = \infty$. Thus the limit is still indeterminate: $0\cdot\infty$. L'Hospital's rule only applies to indeterminate fractions ($0/0$ or ∞/∞), but it is immediately clear how to make this new problem into a fraction: rewrite it as

$$\lim_{x\to\infty}\frac{\ln(\sinh x)}{x}$$

Now it is of the indeterminate for $\frac{\infty}{\infty}$, and so L'Hospital's rule applies:

$$\lim_{x\to\infty}\frac{\ln(\sinh x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{\sinh x}\cosh x}{1} = \lim_{x\to\infty}\frac{\cosh x}{\sinh x}$$

We have used the chain rule to differentiate the numerator.

There are two good ways to go from here. (One should not just say "$\cosh\infty = \infty$ and $\sinh\infty = \infty$, so this is $\frac{\infty}{\infty}$, and so I can use L'Hospital's rule again", because it simply returns $\sinh x/\cosh x$, which is no better. This is why there is the hint.)

One good way to proceed, though, is to recognize $\cosh x/\sinh x = 1/\tanh x$, and so recall that $\lim\limits_{x\to\infty}\tanh x = 1$. Thus

$$\lim_{x\to\infty}\frac{\cosh x}{\sinh x} = \lim_{x\to\infty}\frac{1}{\tanh x} \overset{\text{LL}}{=} \frac{1}{\lim\limits_{x\to\infty}\tanh x} = \frac{1}{1} = 1$$

Another possibility is to write out the definitions of cosh and sinh:

$$\lim_{x\to\infty}\frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to\infty}\frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1+0}{1-0} = 1$$

In either case, we see that the answer to the NEW PROBLEM is 1. But the first manipulation shows that the answer to the old problem is $e^{\text{NEW PROBLEM}}$, and so the final answer is

$$e^1 = \boxed{e}$$