Math 1A: Quiz 10

ANSWERS

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Wednesday, 8 April 2009

You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

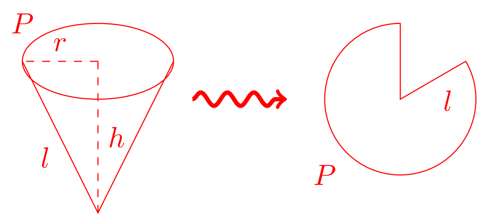
1. (10 pts) A cone-shaped paper drinking cup is to be made to hold $27\,\mathrm{cm}^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.

   At the beginning of the quiz, we gave out two useful facts:

   (a) For any cone-like object, Volume $= \dfrac{1}{3}$ (Area of base) (Height).

   (b) For a wedge of a circle, Area $= \dfrac{1}{2}$ (Radius) (Length of circular part of perimeter).

   We draw a diagram of the problem, and label everything. We have a paper cone, which we can unroll into a wedge of a circle:



   Thus, the volume of the cone is $V = \frac{1}{3}\pi r^2 h$, the area of paper is $A = \frac{1}{2}Pl$, and the perimeter P is $P = 2\pi r$. So the area is $A = \pi r l$, and the relationship between l, r, h is given by the Pythagorean formula $l^2 = r^2 + h^2$.

   For this problem, the volume $V = \frac{1}{3}\pi r^2 h$ is fixed at 27 cubic centimeters — we drop the units. So we are trying to minimize $A = \pi r l$, subject to $\pi r^2 h = 27$ and $l = \sqrt{r^2 + h^2}$. Thus $h = 81/(\pi r^2)$, and $l = \sqrt{r^2 + (81/\pi r^2)^2}$, and so:

$$A(r) = \pi r \sqrt{r^2 + \left(\frac{81}{\pi r^2}\right)^2} = \sqrt{\pi^2 r^4 + \frac{81^2}{r^2}}$$

   As $r \to 0$ or as $r \to \infty$, $A(r) \to +\infty$, and since $A(r)$ is continuous, it must take a minimum at a critical point. We can differentiate A to find a critical point, or we can square A first and then differentiate. Taking the most direct route, we solve:

$$0 = A'(r) = \frac{4\pi^2 r^3 - 2 \cdot 81^2 r^{-3}}{2\sqrt{\pi^2 r^4 + 81^2 r^{-2}}}$$

   Thus, a 0 occurs when the top (equal to the derivative of $(A(r))^2$) is zero. I.e.:

$$0 = 4\pi^2 r^3 - 2 \cdot 81 r^{-3}, \text{ and so } 81^2 r^{-3} = 2\pi^2 r^3$$

   where we added the term in r^{-3} to both sides and divided everything by 2. Then, dividing by $2\pi^2 r^{-3}$, we have

$$\frac{81^2}{2\pi^2} = r^6, \text{ and so } \boxed{r = \sqrt[6]{\frac{81^2}{2\pi^2}} = \sqrt[3]{\frac{81}{\pi\sqrt{2}}}}$$

   Plugging in to $h = 81/(\pi r^2)$ gives $\boxed{h = \dfrac{81}{\pi \sqrt[3]{\frac{81^2}{2\pi^2}}} = \sqrt[3]{\dfrac{2 \cdot 81}{\pi}}}$. We see that $h = \sqrt{2}\, r$.