Math 1A: Quiz 11 ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) Provide an algorithm to calculate the root of $0 = x^5 - 80x + 200$ using Newton's Method. You must:

   (a) Sketch a graph of $y = x^5 - 80x + 200$. Your graph can be rough, but it should be accurate enough to suggest a good starting value x_1 for Newton's Method.

   (b) Specify the formula to calculate x_{n+1} in terms of x_n.

   (a) A good step for sketching is to looks for critical points. The derivative is $y' = 5x^4 - 80 = 5(x^4 - 16)$, so the critical points occur at $x = \pm 2$. Plugging these in, we see that $y(2) = 32 - 160 + 200 = 72$ and $y(-2) = -32 + 160 + 200 = 328$, so the graph of $y = x^5 - 80x + 200$ should have a local minimum at $(2, 72)$ and a local max at $(-2, 328)$. The y-intercept is $(0, 200)$, and the function goes to $+\infty$ in the positive direction and $-\infty$ in the negative direction. Thus, there is a unique real root, and its value is $x < 2$. We predict that $\boxed{x_1 = -3}$ is a good starting value for Newton's Method.

   (b) Newton's method says that for any function $y(x)$, we should estimate using $x_{n+1} = x_n - \dfrac{y(x_n)}{y'(x_n)}$. Evaluating these, we see that we should use:

$$\boxed{x_{n+1} = x_n - \frac{(x_n)^5 - 80x_n + 200}{5(x_n)^4 - 80}}$$

2. (bonus) Calculate x_3 using your answers above (and without the aid of any electronics). Round your answer to the nearest 0.01.

   Starting at $x_1 = -3$, we have

$$x_2 = x_1 - \frac{(x_1)^5 - 80x_1 + 200}{5(x_1)^4 - 80} = -3 - \frac{(-3)^5 - 80(-3) + 200}{5(-3)^4 - 80}$$

   We have $3^5 = 243$ and $3^4 = 81$, so:

$$x_2 = -3 - \frac{-243 + 240 - 200}{5(81) - 80} = -3 - \frac{-203}{325}$$

   We estimate $203/325 \approx 2/3$. More accurately, $200/325 = 8/13 = 0.61(5)$, and $3/325 \approx 3/300 = 0.01$, so $203/325 \approx 0.62(5)$, and going out one more digit shows that this rounds down. Thus:

$$x_2 = -3 - (-0.62) = -2.38$$

   We can now calculate

$$x_3 = x_2 - \frac{(x_2)^5 - 80x_2 + 200}{5(x_2)^4 - 80} = -2.38 - \frac{(-2.38)^5 - 80(-2.38) + 200}{5(2.38)^4 - 80}$$

   Using $(2.4)^4 = (5.76)^2 = 33.1776$, we have $(2.4)^5 = (2.4)(33.18) = 79.63\ldots$. Also $80(2.4) = 192$, and so we use linearization:

$$\begin{aligned} x_3 &\approx -2.38 - \frac{-\big((2.4)^5 + 5(2.4)^4(-0.02)\big) - 80(-2.4) - 80(0.02) + 200}{5(2.4)^4 + 20(2.4)^3(-0.02) - 80} \\ &\approx -2.38 - \frac{-\big(79.63 - 3.32\big) + 192 - 1.60 + 200}{165.89 + 5.53 - 80} = -2.38 - \frac{314.09}{91.42} \end{aligned}$$

   We do linearization again, on $\frac{314}{91} = \frac{314}{100-9} \approx \frac{314}{100} - \frac{314}{100^2}(-9) = 3.14 + (0.0314)(9) = 3.14 + 0.28(3) = 3.42$. So we get $\boxed{x_3 \approx -2.38 - 3.42 = -5.80}$.

   We remark that Newton's Method has not yet started to converge. As such, this by-hand stuff if not particularly valuable, beyond as arithmetic practice.