Math 1A: Quiz 12 ANSWERS

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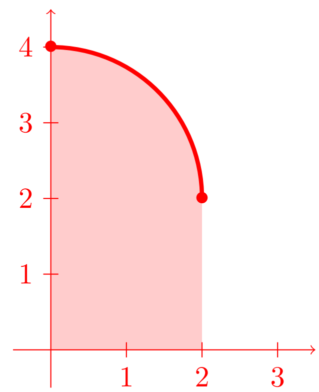
You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (4 pts) Evaluate $\int_0^2 \left(2+\sqrt{4-x^2}\right)\,dx$ by interpreting the definite integral as an area. Do not use the Fundamental Theorem of Calculus.

   We first sketch the corresponding function, and shade the area below it:



   From this, we see that the area consists of a square of sidelength 2 and a quarter circle with radius 2. Recalling that the area of a circle is πr^2 and hence the area of a quarter circle is $\pi r^2/4$, we see that the total area and hence the total integral is:

$$\int_0^2 \left(2+\sqrt{4-x^2}\right)dx = \frac{\pi 2^2}{4} + 2^2 = \boxed{\pi + 4}$$

2. (6 pts) Find an expression for $\int_1^3 e^x\,dx$ as a limit of sums. Do not evaluate the expression.

   For any n, we can approximate the integral $\int_1^3 e^x\,dx$ as a sum of n rectangles, by dividing the interval $[1,3]$ into n equal pieces $1 = x_0 < x_1 < \cdots < x_n = 3$. Then $x_i - x_{i-1} = \Delta x = 2/n$, and $x_i = 1 + i\Delta x = 1 + 2i/n$. The Riemann sum with right endpoints is $\sum_{i=1}^n f(x_i)\Delta x$, and we take the limit as $n \to \infty$:

$$\boxed{\int_1^3 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^n e^{1+2i/n}\,\frac{2}{n}}$$

3. (bonus) Without using the Fundamental Theorem of Calculus, evaluate the limit from question 2. *First Hint:* You may use the following fact without proof:

$$\sum_{i=1}^{n} a^i = a + a^2 + \cdots + a^n = \frac{a^{n+1} - a}{a - 1} = a\frac{a^n - 1}{a - 1} \text{ for any } a \neq 1$$

Second Hint: You will probably need to use L'Hospital's Rule. When you do, you may find that it's easier to first make a substitution: $u = \frac{1}{n}$, $\lim_{n\to\infty} f(n) = \lim_{u\to 0^+} f(1/u)$.

In part 2, we found the formula $\displaystyle\int_1^3 e^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} e^{1+2i/n}\frac{2}{n}$. We compute this, using the first hint with $a = e^{2/n}$ and making the suggested substitution:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} e^{1+2i/n}\frac{2}{n} &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{1+2i/n} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e\cdot e^{2i/n} \\
&= \lim_{n\to\infty}\frac{2e}{n}\sum_{i=1}^{n}(e^{2/n})^i \\
&= \lim_{n\to\infty}\frac{2e}{n}e^{2/n}\frac{(e^{2/n})^n - 1}{e^{2/n} - 1} \\
&= \lim_{n\to\infty}\frac{2e}{n}e^{2/n}\frac{e^2 - 1}{e^{2/n} - 1} \\
&= 2e(e^2 - 1)\lim_{n\to\infty}\frac{1}{n}e^{2/n}\frac{1}{e^{2/n} - 1} \\
&= 2e(e^2 - 1)\lim_{u\to 0^+} ue^{2u}\frac{1}{e^{2u} - 1} \\
&= 2e(e^2 - 1)\lim_{u\to 0^+}\frac{u}{1 - e^{-2u}} \\
&\overset{\text{L'H}}{=} 2e(e^2 - 1)\lim_{u\to 0^+}\frac{1}{-(-2)e^{-2u}} \\
&= 2e(e^2 - 1)\frac{1}{2e^{2\cdot 0}} \\
&= \frac{2e(e^2 - 1)}{2\cdot 1} \\
&= \boxed{e^3 - e}
\end{aligned}$$