Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Review and Preview

1. (a) What is the definition of a *definite integral*?
   (b) What is the definition of an *indefinite integral*?
   (c) What is the statement of the *Fundamental Theorem of Calculus* (FTC)?

2. (a) § Use the definition of a Riemann sum to approximate $\int_0^2 (x^2 - x)dx$ with four subintervals, taking the sample points to be right endpoints.
   (b) Use FTC to calculate $\int_0^2 (x^2 - x)dx$ exactly. What is the error in your estimation from part (a)?
   (c) Repeat steps (a) and (b) with n subintervals, where n is an arbitrary number. You may want to use the following facts:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad\qquad \sum_{i=1}^{n} i^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$$

   Remark: We will develop approximation methods better than the Right (or Left) Endpoint method later in this course. You've met one already in Math 1A: the Midpoint approximation, which takes the sample points for calculating a Riemann sum to be the midpoints of each subinterval. Particularly ambitious students are invited to repeat steps (a) and (b) with the Midpoint approximation.

3. (a) Use geometry to evaluate $f(x) = \int_0^x \sqrt{1-t^2}\,dt$.
   (b) Check the Fundamental Theorem of Calculus, by showing directly that $f'(x) = \sqrt{1-x^2}$.
   (c) Use the substitution $t = \sin\theta$ to evaluate $f(x)$. You will need the *Pythagorean Theorem* and the *Double Angle Formula*:

$$1 - \sin^2\theta = \cos^2\theta \qquad\qquad \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

   Check that you get the same answer as in part (a).

   Remark: We will develop methods similar to the one in part (c) with which to evaluate many integrals that involve terms like $\sqrt{1-t^2}$.

4. (a) State the *Chain Rule* for differentiation.

(b) Explain how the Chain Rule leads directly (via FTC) to the *Substitution Formula* for integration.

5. (a) What's wrong with the following calculations?

$$\int \sqrt{1+e^x}\, e^x\, dx \underset{u=1+e^x}{=} \int \sqrt{u}\,(u-1)\, du = \int \left(u^{3/2} - u^{1/2}\right) du$$
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1+e^x)^{5/2} - \frac{2}{3}(1+e^x)^{3/2} + C$$

$$\int_0^\pi \cos^2 x \sin x\, dx \underset{u=\cos x}{=} \int_0^\pi u^2\, du = \left.\frac{u^3}{3}\right|_0^\pi = \frac{\pi^3}{3} - \frac{0^3}{0} = \frac{\pi^3}{3}$$

(b) Evaluate the two integrals above correctly.

6. § Use a substitution to evaluate the following integrals:

(a) $\displaystyle\int_0^2 y^2 \sqrt{y^3+1}\, dy$ (b) $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$ (c) $\displaystyle\int_0^1 \sin(3\pi t)\, dt$

(d) $\displaystyle\int \sin \pi t \cos \pi t\, dt$ (e) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$ (f) $\displaystyle\int \frac{\cos(\ln x)}{x}\, dx$

7. (a) § Use an integral to estimate the sum $\displaystyle\sum_{i=1}^{1000} \sqrt{i}$.

(b) Is your estimate an overestimate or an underestimate?

(c) Estimate the error in your answer to part (a). When estimating errors, you should always provide an overestimate rather than an underestimate.

(d) Without a calculator, how could you improve the estimate of the value of the sum?

Remark: In addition to developing techniques through which we can estimate the value of an integral using a sum (as in question 2), we will develop techniques to estimate the values of sums, including infinite sums, using integrals.

8. § Sketch the curves $x + y = 0$ and $x = y^2 + 3y$ and find the enclosed area.

9. § Sketch the curves $y = x^2 + 1$ and $y = 9 - x^2$, and find the volume obtained by rotating the enclosed region around the line $y = -1$.

10. Let $p(x)$ be a cubic function such that the curves $y = p(x)$ and $y = x^2$ intersect when $x = 0$, $x = a$, and $x = b$.

(a) Is the function $p(x)$ uniquely determined by the above condition?

(b) Express the above condition algebraically in terms of the function $p(x) - x^2$.

(c) § The two curves inclose two regions. If these two regions have the same area, how is b related to a? *Remark:* there are various cases here, depending on the order of $0, a, b$. Up to switching the names of a and b and switching $x \mapsto -x$, there are two possibilities: either $0 < a < b$ or $a < 0 < b$. Consider those two cases.