Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Always draw pictures.

Integration by Parts

In 1A, you learned the product rule for differentiation:

$$\frac{d}{dx}[f(x)\,g(x)] = \frac{d}{dx}[f(x)]\,g(x) + f(x)\,\frac{d}{dx}[g(x)]$$

This is often abbreviated $(fg)' = f'g + g'f$, or $d(fg) = f\,dg + g\,df$. By the fundamental theorem of calculus, the integral of the derivative of a function is the original function. Thus $fg + C = \int d(fg) = \int [f\,dg + g\,df] = \int f\,dg + \int g\,df$. Rearranging the equality gives the *integration by parts formula*, which is a kind of product rule for indefinite integrals:

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx}$$

What about definite integrals? Then $\int_a^b f\,dg + \int_a^b g\,df = \int_a^b d(fg) = [fg]_a^b = f(b)g(b) - f(a)g(a)$:

$$\boxed{\int_a^b f(x)\,g'(x)\,dx = [f(x)\,g(x)]_a^b - \int_a^b g(x)\,f'(x)\,dx}$$

1. (a) Use the integration by parts formula to integrate

$$\int x\,e^x\,dx$$

Hint: let $f(x) = x$ and $g'(x) = e^x$. What is $g(x)$?

(b) Based on your answer to part (a), find $\int x^2\,e^x\,dx$. Based on that, find $\int x^3\,e^x\,dx$ and $\int x^4\,e^x\,dx$.

(c) Guess the pattern from part (b). Prove your pattern: use integration-by-parts to write $\int x^n\,e^x\,dx$ in terms of $\int x^{n-1}\,e^x\,dx$. This is an example of a *reduction formula*.

2. (a) Use the integration by parts formula to integrate

$$\int \ln x\,dx$$

Hint: let $f(x) = \ln x$ and $g'(x) = 1$.

(b) Use the integration by parts formula to integrate

$$\int (\ln x)^2 \, dx$$

Hint: let $f(x) = (\ln x)^2$ and $g'(x) = 1$.

(c) Use the integration by parts formula to integrate

$$\int (\ln x)^n \, dx$$

3. Find a formula for $\int f(x)\, g''(x)\, dx$ by applying the integration-by-parts formula twice. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$ (and that $f''(x)$ is continuous on $[1, 4]$). What is $\int_1^4 x\, f''(x)\, dx$?

4. By integrating by parts twice and rearranging, find $\int e^x \cos x \, dx$. What is $\int e^x \sin x \, dx$? How about $\int e^{ax} \cos x \, dx$, where a is a constant? How about $\int x\, e^x \cos x \, dx$?

5. § Use integration by parts to evaluate the following integrals. For some you will first need to make a substitution.

(a) $\displaystyle\int t \sin 2t \, dt$ (b) $\displaystyle\int x^2 \sin \pi x \, dx$ (c) $\displaystyle\int \arcsin x \, dx$

(d) $\displaystyle\int s\, 2^s \, ds$ (e) $\displaystyle\int_0^1 (x^2 + 1)e^{-x} \, dx$ (f) $\displaystyle\int_4^9 \frac{\ln y}{\sqrt{y}} \, dy$

(g) $\displaystyle\int_0^{\pi} x^3 \cos x \, dx$ (h) $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x) \, dx$ (i) $\displaystyle\int_0^t e^s \sin(t - s) \, ds$

(j) $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) \, d\theta$ (k) $\displaystyle\int_0^{\pi} e^{\cos t} \sin 2t \, dt$ (l) $\displaystyle\int \sin(\ln x) \, dx$

6. What's wrong with the following proof that $0 = 1$?

$$\ln x = \int \frac{1}{x} \, dx = \frac{1}{x}\, x - \int \frac{-1}{x^2}\, x \, dx = 1 + \int \frac{1}{x} \, dx = 1 + \ln x$$

7. Why can we forget to add an arbitrary constant during the intermediate steps when integrating by parts? Integrate $\int x^n e^x dx$ completely honestly: let $u = x^n$ and $dv = e^x dx$, but this time let $v = e^x + C$.

8. Let $n = 2k + 1$ be an odd integer. Calculate $\int_{x=0}^{\pi/2} \cos^n x \, dx$ in two different ways:

(a) Using a reduction formula. What happens to the boundary terms (the uv in $\int u \, dv = uv - \int v \, du$)? Why does it matter that n is odd? (The formula is different for even n.)

(b) Using a u-substitution. (Hint: $\cos^2 x = 1 - \sin^2 x$, and $\sin' x = \cos x$.) For any given k, you could then expand out and evaluate the integral. For a general k, you can use integration by parts to get a reduction formula.

9. What is $\int_0^{\pi/2} dx$? Use the reduction formula from the previous problem to evaluate $\int_0^{\pi/2} \cos^n x \, dx$ for n even.