Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Trigonometric Substitutions

Many mathematical formulae involve the square roots of sums or differences of squares. Any integral involving, say, $\sqrt{a^2 \pm x^2}$, should make you think of the Pythagorean theorem, and hence trigonometry. The following versions of the Pythagorean formula are especially useful:

Pythagorean identity	Suggested substitution	Part of integral	$dx =$
$1 - \sin^2\theta = \cos^2\theta$	$x = a\sin\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$	$dx = a\cos\theta\,d\theta$
$-1 + \sec^2\theta = \tan^2\theta$	$x = a\sec\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$	$dx = a\tan\theta\sec\theta\,d\theta$
$1 + \tan^2\theta = \sec^2\theta$	$x = a\tan\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$	$dx = a\sec^2\theta\,d\theta$

We always take θ in the range $0 \leq \theta \leq \pi/2$, so that all trig functions are positive. Feel free to change the bounds of integration, but attend to the signs of all terms and the domain in θ. The last integral is also particularly useful when the integral includes $1/(a^2 + x^2)$, even without a square root.

1. § Try these integrals:

   (a) $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$ (b) $\displaystyle\int \frac{x^3}{\sqrt{x^2+100}}\,dx$ (c) $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\,dx$

   (d) $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx$ (e) $\displaystyle\int \frac{du}{u\sqrt{5-u^2}}$ (f) $\displaystyle\int \frac{dt}{\sqrt{25-t^2}}$

   (g) $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$ (h) $\displaystyle\int \frac{dx}{\big((ax)^2-b^2\big)^{3/2}}$ (i) $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

2. Evaluate the integral $\displaystyle\int \frac{x^n}{1+x^2}\,dx$. Your answer should depend on whether n is even or odd.

3. (a) Graph the function $y = x^2 + 2x + 2$. What is the domain of the function $\sqrt{x^2+2x+2}$? Find numbers r and a such that $x^2 + 2x + 2 = (x-r)^2 \pm a^2$.

   (b) Evaluate the following integrals:

   $$\int \sqrt{x^2+2x+2}\,dx \qquad \int \frac{dx}{\sqrt{x^2+2x+2}} \qquad \int \frac{dx}{(x^2+2x+2)^2}$$

4. (a) Graph the function $y = x^2 + 4x - 5$. What is the domain of the function $\sqrt{x^2 + 4x - 5}$? Find numbers r and a such that $x^2 + 4x - 5 = (x - r)^2 \pm a^2$.
   (b) Evaluate the following integrals:

$$\int \sqrt{x^2 + 4x - 5}\, dx \qquad\qquad \int \frac{dx}{\sqrt{x^2 + 4x - 5}}$$

5. § Find the average value of $f(x) = \sqrt{x^2 - 1}/x$ for $1 \le x \le 7$.

6. Draw the ellipses and find their areas:
   (a) $25x^2 + 9y^2 - 100x + 18y - 116 = 0$
   (b) $13x^2 + 13y^2 + 10xy = 25$

7. Imagine taking a solid sphere of radius 1, and slicing it by a plane slice a distance $a < 1$ from the center. What are the volumes of the two pieces?

8. § You're standing on a pier. There's a boat in the water at distance L from you, connected to a rope of length L, and you're holding the other end (the rope is completely taut). Imagine that you start to walk along the pier, pulling the rope; sketch the path the boat follows.

   In fact, the boat will follow a path called a *tractrix*; it's defined by the property that the rope is always tangent to the path of the boat. To find an equation for the path as a function $y = y(x)$, solve the following integral equation:

$$\int \frac{dy}{y} = \int \frac{-\sqrt{L^2 - x^2}}{x} dx$$

9. (a) § Evaluate $\int \frac{dx}{\sqrt{x^2+1}}$ in two different ways: (i) use a trigonometric substitution; (ii) use the substitution $x = \sinh t = \frac{e^t - e^{-t}}{2}$.
   (b) Solve the equation $x = \frac{e^t - e^{-t}}{2}$ for t by multiplying both sides by e^t, solving a quadratic equation, and taking natural logarithms; by doing this, you will find a formula for $\operatorname{arcsinh} x$. Use this formula to prove that your answers in part (a) are the same.
   (c) What happens with $\int \frac{dx}{\sqrt{x^2+a^2}}$? Solve it in two different ways, and use your formula for arcsinh to check that your answers are the same.