Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Partial Fraction Decomposition

When we add fractions, we go through all sorts of steps to find a common denominator, because for some purposes it's best to have an expressing written as a single fraction rather than as multiple fractions:

$$\frac{1}{2x+1} + \frac{x}{x^2+1} = \frac{x^2+x+1}{2x^3+x^2+2x+1}$$

But for integration, complicated fractions like $(x^2+x+1)/(2x^3+x^2+2x+1)$ are less than useful, whereas the simpler fractions are perfectly tractable:

$$\int \frac{x^2+x+1}{2x^3+x^2+2x+1}\,dx = \int \left(\frac{1}{2x+1} + \frac{x}{x^2+1}\right)dx = \frac{1}{2}\ln(2x+1) + \frac{1}{2}\ln(x^2+1) + C$$

(We compute the second integral by substituting $u = x^2+1$.)

In fact, every rational function (the ratio of two polynomials) can be decomposed uniquely into a sum of (a polynomial plus) simple fractions. A "simple fraction" is one where the denominator is a power of a linear or irreducible-quadratic polynomial, and the numerator is of lower degree than the root of the denominator.

The zeroth step is to perform long division so that your fraction is written as a "mixed number": a polynomial plus a fraction in which the numerator has lower degree than the denominator. The first step is to factor the denominator completely: $2x^3+x^2+2x+1 = (x^2+1)(2x+1)$. Over the real numbers, every polynomial has a unique factorization into linear and irreducible-quadratic parts, but finding such a factorization in general can be very hard.

Then each factor in the denominator corresponds to a potential term in the decomposition, with repeated factors counting multiply. Write the most general decomposition given the factorization, and then solve for the unknown coefficients.[1]

1. Decompose the following fractions into sums of simple fractions:

   (a) $\dfrac{x}{(x+2)(x-1)}$ (b) $\dfrac{x-1}{x^3+x}$ (c) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2}$

[1] For the more curious of you, a few years ago (back when I kept a semi-regular blog) I wrote a very short proof that partial fraction decompositions always exist: `http://theojf.blogspot.com/2007/09/partial-fractions.html`.

2. § Compute the following integrals. Remember that $\int dx/(x^2+1) = \arctan x$ and that $\int 2x\,dx/(x^2+1) = \ln(x^2+1)$. If necessary, don't forget to complete the square in the denominator:

(a) $\displaystyle\int \frac{x-1}{x^2+3x+2}\,dx$ (b) $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}\,dx$ (c) $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

(d) $\displaystyle\int \frac{3x^2+x+4}{x^4+3x^2+1}\,dx$ (e) $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx$ (f) $\displaystyle\int \frac{x^3}{x^3+1}\,dx$

3. Solve $\int x/(x^2-1)\,dx$ (a) with a u-substitution, (b) with a trig substitution, (c) with a partial-fractions decomposition.

4. Evaluate $\int \sec\theta\,d\theta$ as follows: write the integrand in terms of sines and cosines; multiply the numerator and denominator by $\cos\theta$; use trig identities to rewrite the denominator in terms of $\sin\theta$; make a u-substitution; decompose the resulting integrand into partial fractions; integrate and substitute back for θ.

5. (a) Compute $\int \tan^2\theta\,\sec\theta\,d\theta$ by first performing a u-substitution and then decomposing into partial fractions.

   (b) Compute $\int (x^2+4)^{3/2}\,dx$ by first using a trig sub, then a u-sub, then partial fractions.

6. § Evaluate $\displaystyle\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$. Hint: substitute $u = \sqrt[6]{x}$.

7. § Factor x^4+1 as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4+1)\,dx$.

8. Any rational expression in sines and cosines can be made into a rational function via the *Weierstrass substitution*: $u = \tan(\theta/2)$.

   (a) Using trig identities, find general formulae for $\sin\theta$, $\cos\theta$, and $d\theta$ in terms of u.

   (b) § Compute (i) $\displaystyle\int \frac{d\theta}{3\cos\theta - 4\sin\theta}$ and (ii) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{d\theta}{1+\sin\theta-\cos\theta}$.

9. § If f is a quadratic function such that $f(0)=1$ and $\displaystyle\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function (no ln and arctan terms), find the value of $f'(0)$.

10. **This problem is only for those who know complex numbers.** Normally we integrate $\int dx/(x^2+1) = \arctan(x)$ with a trig substitution. Factor the denominator with complex numbers, decompose it as partial fractions, and integrate: what does this say about the relationship between arctan and ln?