Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Approximate Integration — Corrected

To approximate $\int_a^b f(x)\,dx$, let $\Delta x = (b-a)/n$, $x_i = a + i\Delta x$, and $\bar{x}_i = (x_{i-1} + x_i)/2$. Then define the following approximations (we use different number from Stewart for Simpson's Rule S_n):

$$L_n = (f(x_0) + \cdots + f(x_{n-1}))\,\Delta x \qquad R_n = (f(x_1) + \cdots + f(x_n))\,\Delta x$$

$$T_n = (f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n))\,\frac{\Delta x}{2} \qquad M_n = (f(\bar{x}_1) + \cdots + f(\bar{x}_n))\,\Delta x$$

$$S_n = (f(x_0) + 4f(\bar{x}_1) + 2f(x_1) + 4f(\bar{x}_2) + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(\bar{x}_n) + f(x_n))\,\frac{\Delta x}{6}$$

These have the following errors:

$$|E_L| \leq \sup_{x\in[a,b]} \left|f'(x)\right| \frac{(b-a)^2}{2n} \qquad |E_R| \leq \sup_{x\in[a,b]} \left|f'(x)\right| \frac{(b-a)^2}{2n}$$

$$|E_T| \leq \sup_{x\in[a,b]} \left|f''(x)\right| \frac{(b-a)^3}{12n^2} \qquad |E_M| \leq \sup_{x\in[a,b]} \left|f''(x)\right| \frac{(b-a)^3}{24n^2}$$

$$|E_S| \leq \sup_{x\in[a,b]} \left|f^{(4)}(x)\right| \frac{(b-a)^5}{2880n^4}$$

The word "sup" is short for "supremum" — the symbol "$\sup_{x\in[a,b]} g(x)$" means "the largest value of $g(x)$ as x ranges over $[a,b]$". In practice, it suffices to replace the suprema with some easy-to-compute numbers which are even bigger.

1. Explain each of the above approximation techniques when $n = 1$.

2. Let $f(x)$ be a positive increasing function with negative second derivative on $[a,b]$. Place the following five numbers in increasing order: L_n, R_n, T_n, M_n, and $\int_a^b f(x)dx$.

3. Of the Midpoint and Trapezoid rules, pick your favorite. If you were to evaluate each of the following integrals using that rule with 5 subintervals, what would be your expected error?

How many subintervals would you need to ensure an error less than 0.00001?

(a) $\int_1^4 \sqrt{1+\sqrt{x}}\,dx$ (b) $\int_1^2 \sqrt{z}\,e^{-z}\,dz$ (c) $\int_4^6 \ln(x^3+2)\,dx$

4. (a) Show that $(L_n + R_n)/2 = T_n$.
   (b) § Show that $(T_n + M_n)/2 = T_{2n}$.
   (c) § Show that $(T_n + 2M_n)/3 = S_n$.

5. By explicit calculation, show that Simpson's rule calculates the area under a cubic curve exactly. What are the highest degree polynomials the rest of the approximation rules calculate exactly?

6. By explicit calculation, show that the errors for L_n and R_n are exact when $f(x)$ is a linear function, and that the errors for T_n and M_n are exact when $f(x)$ is a quadratic function.

7. Make sense of the following proof from *Proofs without Words: Exercises in Visual Thinking* by Roger B. Nelsen (1993):