Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Improper Integrals

1. (a) Find all values of p for which $\int_1^\infty x^p\,dx$ converges. If $\int x^p\,dx$ converges, to what does it converge? (Your answer should, of course, be a function of p.)
   (b) Find all values of p for which $\int_0^1 x^p\,dx$ converges. If $\int_0^1 x^p\,dx$ converges, to what does it converge? (Your answer should, of course, be a function of p.)
   (c) Are there any values of p for which $\int_0^\infty x^p\,dx$ converges?
2. Let n be a nonnegative integer. Using integration by parts, find $\int_0^\infty x^n\,e^{-x}\,dx$.
3. Show that $\int_0^1 \ln x\,dx$ converges, and find the limit. More generally, use integration by parts to find $\int_0^1 (\ln x)^n\,dx$ for any nonnegative integer n.
4. Is $\int_0^\pi \tan x\,dx$ well-defined (i.e. does the integral converge)? If so, to what?
5. § The "Laplace Transform" of a function $f(t)$, is the function of s given by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

   if this integral converges. Find the Laplace transforms $\mathcal{L}[1](s)$, $\mathcal{L}[t](s)$, $\mathcal{L}[e^t](s)$, and $\mathcal{L}[t^n](s)$. (The last one requires your answer to the previous question.) What are the domains of these functions (for what s values do the integrals converge)?

A Comparison Test

If $0 \le f(x) \le g(x)$ on $[a,b]$ and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ also converges, and $0 \le \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. The functions are allowed to have (certain kinds of, but that's a technical issue that we will ignore in this course) discontinuities, and a and b are allowed to be $\pm\infty$.

1. Use the comparison test to decide if the following integrals converge:

(a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ (b) $\displaystyle\int_1^\infty \frac{4+e^{-x}}{x+1}\,dx$ (c) $\displaystyle\int_0^{\pi/2} \frac{dx}{x\sin x}$ (d) $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\,dx$

2. Evaluate the integral or show that it is divergent:

(a) $\displaystyle\int_0^1 \frac{t^2+1}{t^2-1}\,dt$ (b) $\displaystyle\int_2^6 \frac{y\,dy}{\sqrt{y-2}}$ (c) $\displaystyle\int_0^1 \frac{dx}{2-3x}$ (d) $\displaystyle\int_{-1}^1 \frac{x+1}{\sqrt[3]{x^4}}\,dx$

3. In terms of b and c, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+bx+c}.$$

What conditions do you need to place on b and c to assure that this integral converges?

4. § Find the value of the constant C for which the integral

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx$$

converges. Evaluate the integral for this value of C.