Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Arc Length

The length of the curve $\gamma = \{y = f(x) : a \leq x \leq b\}$ is given by the formula

$$\boxed{\ell(\gamma) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

For a curve of the form $x = g(y)$, we also have the arclength $\int \sqrt{1 + (g'(y))}\, dy$. In general, the arc length is $\int ds$, where $ds = \sqrt{dx^2 + dy^2}$.

1. Find the lengths of the curves:

   (a) $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}, \quad 2 \leq x \leq 4$ (b) $y^2 = 4x, \quad 0 \leq y \leq 2$

2. Using the arc length formula, prove the formula for the circumference of a circle.

3. Set up, but do not evaluate, an integral for the perimeter of the ellipse $\{x^2/a^2 + y^2/b^2 = 1\}$. (Evaluating this integral is notoriously hard; integrals of this form are called, not surprisingly, "elliptical integrals.")

4. § (a) Sketch the curve $\{y^3 = x^2\}$.
   (b) Set up two integrals, one in terms of x and one in terms of y, for the arc length of the above curve from $(0, 0)$ to $(1, 1)$. One of your integrals should be an improper integral. Evaluate each of them.
   (c) Find the length of the arc of this curve from $(-1, 1)$ to $(8, 4)$.

Surface Area of a surface of revolution

When the curve $\gamma = \{y = f(x) : a \leq x \leq b\}$ is rotated around the x-axis, the surface area of the surface of revolution is

$$\boxed{\mathcal{A}_x[\gamma] = 2\pi \int_a^b f(x)\, \sqrt{1 + (f'(x))^2}\, dx}$$

If the same curve is rotated around the y-axis, the surface area if

$$\boxed{\mathcal{A}_y[\gamma] = 2\pi \int_a^b x\sqrt{1+(f'(x))^2}\,dx}$$

1. Find the areas traced out by rotating the curve around the x-axis:

   (a) $y = \cos 2x, \quad 0 \leq x \leq \pi/6$

   (b) $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad \dfrac{1}{2} \leq x \leq 1$

   (c) $y = \ln(x^2+1), \quad 0 \leq x \leq 1$

   (d) $y = e^{2x}, \quad 0 \leq x \leq 1$

2. § Find the surface area of the ellipsoid formed by rotating $\{x^2/a^2 + y^2/b^2 = 1\}$ (where $a > b$) around the x-axis.

3. Find the surface area of the torus formed by rotating $\{(x-3)^2 + y^2 = 1\}$ around the y-axis.

4. Consider a sphere of radius 1 centered at the origin, sliced by two planes $x = a$ and $x = b$, $b > a$. Find the surface area between the two slices.

5. Consider infinite region above the x-axis, to the right of $x = 1$, and below the curve $y = 1/x$. Rotate this region around the x-axis. Show that the volume of revolution has infinite surface area but finite volume: If you fill this “cone” with paint, it will only hold finitely much, but it takes infinitely much paint to coat the inside wall.

6. Let f be a positive differentiable function on $[0,1]$. In the notation above, show that:

$$\mathcal{A}_x[f+1] = \mathcal{A}_x[f] + 2\pi\ell[f]$$