Math 1B: Discussion Exercises

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Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Applications of Integration — Physical Geometry

Stewart describes the following applications of integration to physics and engineering:

- The pressure exerted by a fluid with density ρ at depth h is $P = \rho g h$, where $g = 10\,\mathrm{m/s^2}$ is the acceleration due to gravity. Thus, the total force on a flat plate is:

$$F = \int_a^b \rho(x) g\, x\, f(x)\, dx$$

  where $\rho(x)$ is the density at depth x (for liquids under normal conditions $\rho(x) = \rho$ is constant), and $f(x)$ is the width of the plate at depth x (the plate lies between depths a and b, with $0 \le a \le b$ — we measure depths down from the surface).

- The *center of mass* of a distribution of masses is a "weighted average" of the locations of the masses, each mass "weighted" in the average by its mass (in fact, this is why such averages have are called as such). For masses continuously distributed in one dimension, so that the linear density at location x is $\rho(x)$ (with $\rho(x) = 0$ outside the interval $x \in [a, b]$), the center of mass is given by:

$$\bar{x} = \frac{\int_a^b x\, \rho(x)\, dx}{\int_a^b \rho(x)\, dx}$$

  The denominator is the *total mass* m of the distribution. The *moment of the distribution* is $m\bar{x}$.

- If the masses are distributed in two or more dimensions, the moments and centers of mass may be computed dimension-by-dimension to give coordinates. The *centroid* of a region is the center of mass of the region, where the region is given constant density. If a region with area A is bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$, where $a \le b$ and $f(x) \le g(x)$ for $x \in [a, b]$, then the centroid is located at $(\bar{x}, \bar{y})$ where:

$$\bar{x} = \frac{1}{A}\int_a^b x\,[g(x) - f(x)]\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(g(x))^2 - (f(x))^2\right] dx$$

  Of course, $A = \int_a^b [g(x) - f(x)]\, dx$.

Here's one more physics definitions:

- The *angular moment of inertia* of a one-dimensional object with linear density $\rho(x)$ (supported on the interval $[a, b]$) is given by $\int_a^b x^2\, \rho(x)\, dx$.

Here are some math problems that use the above notions:

1. § A vertical plate in the shape of an equilateral triangle with sidelength $2\,\mathrm{m}$ is submerged in water (density $\rho = 1\,\mathrm{g/cm^3}$) such that one edge is touching the surface of the water. How much pressure is applied to one side of the plate? Be careful with units.

2. If the plate in the previous problem is rotated $180°$ so that its upper point touches the surface of the water, what is the total pressure applied to one side of the plate?

3. Prove that the pressure applied to one side of a plate submerged vertically in water depends only on the area of the plate (or rather of the part of the plate actually under the water) and on the depth of its centroid.

4. § A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, estimate the pressure on each of the five sides of the pool.

5. Prove the following theorem of Archimedes: an object fully submerged in a fluid experiences an upward "buoyant" force equal to the weight of the fluid that would fill the volume of the object. Hint: consider first a normally-oriented rectangular box (you can prove the theorem for boxes without calculus). Then use integral-style arguments to prove the theorem for arbitrary objects.

6. Sketch the region bounded by the given curves and find the centroid:
   (a) § $y = e^x$, $y = 0$, $x = 0$, $x = 1$
   (b) § $y = x^2$, $x = y^2$
   (c) § $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$
   (d) $y = e^x$, $y = 0$, $x = 0$

7. Find the moment and angular moment of a circle (with constant density ρ) of radius r located at $(a, 0)$.

8. Consider a one-dimensional distribution with mass m, moment M, and angular moment I. If you move the object one unit to the right, what happens to the mass? The moment? The angular moment?

9. Consider two distributions with equal total mass. Prove that the center of mass of the combined distribution is halfway between the centers of masses of the separate distributions. What happens when the distributions have different masses?

10. Prove that the moment of a sum of two distributions is the sum of the moments of the separate distributions.

11. Prove that the centroid of any triangle is located at the point of intersection of the medians.

12. Recall that when the curve $y = f(x)$ is rotated around the x-axis, the surface area of the piece of curve corresponding to the interval $[x, x + dx]$ is $dA = 2\pi f(x)\,ds$, where $ds = \sqrt{1 + (f'(x))^2}\,dx$. Find the total hydrostatic force felt by a sphere of radius $1\,\mathrm{m}$ submerged under water so that the center is at a depth h (with $h \geq 1\,\mathrm{m}$). Hint: orient the axes with x pointing down through the center of the sphere and $x = 0$ corresponding to the surface of the water.