Math 1B: Discussion Exercises

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Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Even More Applications of Integration

The velocity v of a liquid flowing at distance r from the center of a cylindrical channel of radius R is given by $v(r) = \frac{1}{4}\frac{P}{l}\frac{1}{\eta}(R^2 - r^2)$, where η is the viscosity of the liquid, P is the pressure, and l is the length of the channel. (The fraction P/l is the pressure per unit length. It's not surprising that this quantity determines the velocity; what's surprising is the sensitivity to the radius.) The *flux* through a channel is the amount of fluid across a cross-section of the channel per unit time. We integrate $\int_{r=0}^{R} v\,dA$, where $dA = 2\pi r\,dr$ is the infinitesimal area at radius r, to conclude that the flux in a cylindrical channel is $\phi = \frac{\pi}{8}\frac{P}{l}\frac{1}{\eta}R^4$. [We use ϕ for "flux", so that we can save the letter F for "force".]

Another extremely important use of integration is in the definition of *work*. If an object moves a distance x against a force F, the work done on the object is $W = Fx$. If the force varies with location, so that $F = F(x)$, then the work is $W = \int_a^b F(x)dx$, where the object moves from location a to location b.

1. (a) Prove the following theorem of Archimedes: an object fully submerged in a fluid experiences an upward "buoyant" force equal to the weight of the fluid that would fill the volume of the object. Hint: consider first a normally-oriented rectangular box (you can prove the theorem for boxes without calculus). Then use integral-style arguments to prove the theorem for arbitrary objects.

   (b) What is the work required to push an object with volume V a distance h down under water?

2. In a spring with spring constant k, the force at location x is $-kx$. Find the work required to move an object from location a to location b.

3. In a pendulum with mass m and length l, the force required to move the bob up along the arc of the pendulum, when it has already moved a distance x from the bottom of the arc, is given by $F(x) = \frac{1}{2\pi}mg\sin\frac{x}{2\pi l}$. Find the work required to move the mass a distance x along the arc of the pendulum, if it starts at the bottom of the arc.

4. (a) A charged particle at a distance x from another charge feels a force $F(x) = k/x^2$, where k depends on the two charges. Find the work required to move from distance a to distance b. Find the work required to move the charge to distance a from ∞.

   (b) If the electric field is created not be another charge but by an electric dipole, then the force is $F(x) = k/x^3$ for some k. Find the work required to move from distance a to distance b. Find the work required to move the charge to distance a from ∞.

5. Lets say an object moves along the curve $y = f(x)$ for $x \in [a, b]$, against a frictive force $F(x)$. Then the total work performed is $\int_a^b F(x)\,ds$, where $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (f'(x))^2}\,dx$.

(a) If the friction is $F(x) = x$ and the curve is $y = x^2$, $x \in [0, 1]$, what is the total work?

(b) If the force of friction is constant, what is the relationship between work and arclength?

6. Rather than a circular pipe, let's consider a channel that consists of two horizontal planes separated at distance H — really we're considering a pipe that consists of a very wide rectangle with height H, and width much much more than H. We consider a liquid flowing through the pipe under laminar conditions. Let h measure the height from the top of the channel; then the velocity $v(h)$ of the liquid at height h satisfies the differential equation

$$\frac{d^2v}{dh^2} = \frac{1}{\eta}\frac{P}{l}$$

where P is the pressure on the channel and l is the length of the channel, so that P/l is the pressure drop per unit length.

(a) We haven't talked about differential equations yet. Nevertheless, we will solve this one. Assume that P is constant throughout the channel, and find the most general function $v(h)$ that satisfies the above differential equation.

(b) The laws of physics require that the velocity of a fluid is zero at the edge of a channel: $v(0) = 0 = v(H)$. Given these conditions and your answer to part (a), find an explicit formula for $v(h)$.

(c) Find the linear flux across across the channel by integrating $\int_0^H v(h)\,dh$. The actual flux is this number times the width of the channel.

(d) Let's now suppose that the force of gravity is quite strong, so that the pressure P depends on the height h, via $P(h) = \rho g h + P_0$, where ρ is the density of the liquid and g is the acceleration due to gravity (P_0 is the pressure applied by the pump). Repeat steps (a) through (c) above with this pressure.

(e) Finally, let's suppose that the "liquid" consists of electrons flowing in a conductor, and that "gravity" is actually an external vertical electric field. Then $P(h) = K/(h+A)+P_0$ for some some numbers K, A, and P_0 (A is essentially the distance from the channel to the external field, and K depends on the strength of the field). Repeat steps (a) through (c) for this pressure.

7. Recall that when the curve $y = f(x)$ is rotated around the x-axis, the surface area of the piece of curve corresponding to the interval $[x, x + dx]$ is $dA = 2\pi f(x)\,ds$, where $ds = \sqrt{1 + (f'(x))^2}\,dx$. Find the total hydrostatic force felt by a sphere of radius 1 m submerged under water so that the center is at a depth h (with $h \geq 1$ m). Hint: orient the axes with x pointing down through the center of the sphere and $x = 0$ corresponding to the surface of the water.

8. **Not a physics problem.** § *Pareto's Law of Income* states that the number of people with incomes between $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}\,dx$ for some constants A and k. It is remarkably accurate for large incomes. The *average income* of these people is $\bar{x} = \frac{1}{N}\int_a^b Ax^{1-k}\,dx$.

(a) There are only finitely many people in the world. For Pareto's Law to be true for large incomes, what conditions must we impose on k?

(b) There is only finitely much money in the world. For Pareto's Law to be true for large incomes, what conditions must we impose on k?