Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Today's exercises are from Rob Bayer.

Intro to Differential Equations

A *differential equation* is an equation relating a function to its derivatives. Differential equations show up in modeling problems, where the rate of change of one variable depends on the value of that variable and perhaps also on others. A *solution* to a differential equation is any function that satisfies the equation — differential equations generally have infinitely many solutions. An *initial value problem* is a differential equation along with a prescribed value of the function in question. Initial value problems generally have unique solutions.

1. For each of the following differential equations, determine if the given function is a solution:

   (a) $y' = e^x + y;\ y = xe^x$

   (b) $\frac{dP}{dt} = 1 + P^2;\ P = \tan x$

   (c) $(y')^2 = 4 + y^2;\ y = e^x - e^{-x}$

   (d) $f' = \frac{1}{e^f};\ f = \ln(x + C)$

2. (a) Find the general solution for each of the following differential equations:

   i. $\frac{dy}{dx} = e^x$

   ii. $y'' = 20x^3 + 2$

   (b) Solve the initial value problem $y' = e^x$, $y(0) = 3$.

   (c) How many initial conditions would you need in order to solve an initial value problem for part ii.?

3. For each of the following cases, write down a di?erential equation that expresses the given idea mathematically:

   (a) A function that equals its own derivative.

   (b) A function that equals its second derivative.

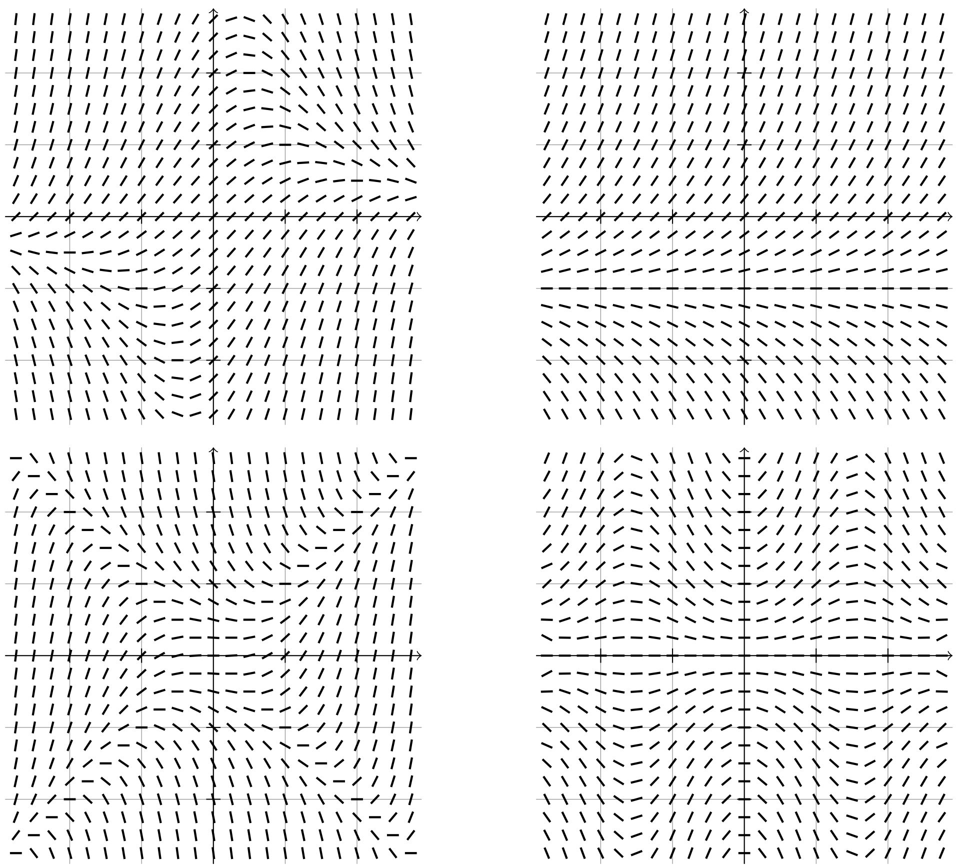
   (c) A function that equals the negative of its second derivative.

4. Show that $y = C\sec x + C\tan x$ is a one-parameter family of solutions to the differential equation $y' = y\sec x$. If $y(\pi/4) = 5$, what is C?

A powerful qualitative tool for studying a differential equation is its *direction field*, which is a "graph" of the differential equation: if $y' = F(x, y)$ is the differential equation in question, then at each point (x, y) in the plane, we draw a small line segment with slope $F(x, y)$. An *equilibrium solution* to a differential equation $y' = F(x, y)$ is a function $y = f(x)$ such that $\lim_{x\to\infty} f(x)$ exists and is finite, so that y is "eventually constant". These occur most easily when the differential equation is of the form $y' = F(y)$, so that it depends on y only; then equilibrium solutions correspond to solutions $0 = F(y)$. A direction field allows for finding equilibrium solutions visually. An equilibrium solution is *stable* if nearby solutions converge towards the equilibrium solution; it is *unstable* if nearby solutions diverge from the equilibrium solution.

5. Suppose a function $y(t)$ satises the differential equation $y' = y^4 - 6y^3 + 5y^2$.
   (a) What are the constant/equilibrium solutions?
   (b) When is y increasing? Decreasing?
   (c) Sketch a direction field for the differential equation, and sketch a few solutions.
   (d) Determine which of the equilibrium solutions are stable or unstable.

6. Consider the following four direction fields:



   (a) Without thinking hardly at all, which one of these is for $y' = 1 + y$? How can you tell?
   (b) The differential equations for the other three are $y' = x^2 - y^2$, $y' = y \sin 2x$, and $y' = 1 - xy$. Determine which is which.
   (c) Using the direction elds, sketch some solution curves to $y' = x^2 - y^2$.

7. Consider the differential equation $y' = x/y$.
   (a) In what regions in the xy-plane is y' positive? Negative? Where is it 0? ∞?
   (b) Sketch the direction eld for this equation.
   (c) Find all solutions of the form $y = mx + b$.