Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Separable Differential Equations

If a differential equation $\frac{dy}{dx} = F(x,y)$ can be written as a product $\frac{dy}{dx} = f(x)g(y)$, where $f(x)$ does not depend on y and $g(y)$ does not depend on x, then the solutions can be found by cross-multiplying and integrating: $\int (g(y))^{-1}dy = \int f(x)dx$. Don't forget to add a constant of integration, and then solve for y in terms of x if you can.

1. § Solve the differential equation:

   (a) $\dfrac{dy}{dx} = \dfrac{y}{x}$ (b) $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$ (c) $(x^2+1)y' = xy$

   (d) $y' = y^2 \sin x$ (e) $(1+\tan y)y' = x^2+1$ (f) $\dfrac{du}{dr} = \dfrac{1+\sqrt{r}}{1+\sqrt{u}}$

2. § Solve the initial value problem:

   (a) $\dfrac{du}{dt} = \dfrac{2t+\sec^2 t}{2u},\ u(0) = -5$ (b) $xy' + y = y^2,\ y(1) = -1$

   (c) $y' \tan x = a + y,\ y(\pi/3) = a,\ 0 < x < \pi/2$ (d) $\dfrac{dL}{dt} = kL^2 \ln t,\ L(1) = -1$

   In (c) and (d), a and k are constants.

3. § Find an equation of the curve that passes through the point $(0,1)$ and whose slope at (x,y) is xy.

4. § Recall that two lines are perpendicular if their slopes are negative reciprocals. Thus, two curves $y = f(x)$ and $y = g(x)$ that intersect at the point (x,y) are perpendicular at that point if and only if $f'(x) = -(g'(x))^{-1}$. Thus, for each of the following families of curves, write a differential equation that a curve must satisfy in order to be perpendicular at every intersection with every member of the family, and then solve the corresponding differential equation:

   (a) $x^2 + 2y^2 = k^2$ (b) $y = \dfrac{k}{x}$ (c) $y = \dfrac{x}{1+kx}$

5. Let n be an arbitrary number. Find the family of curves perpendicular to the family $y = kx^n$. For a few different values of n, graph several members of each family.

6. § When chemists consider a reaction, say A+B → C, they write [A] (etc.) for the concentration at time t of reactant A. For example, consider the reaction $H_2 + B_2 \to 2HBr$. Experiments show that this reaction satisfies the rate law

$$\frac{d[\mathrm{HBr}]}{dt} = k[\mathrm{H_2}][\mathrm{Br_2}]^{1/2}$$

when [HBr] is not too big.

   (a) In mathematics notation, let x be the concentration of HBr, which we assume to start at $x(0) = 0$. If a and b are the initial concentrations of H_2 and Br_2, explain why the above equation is equivalent to:

$$\frac{dx}{dt} = k(a - x)\sqrt{b - x}$$

   (b) Assume that $a = b$. Find $x(t)$ such that $x(0) = 0$.

   (c) Assume that $a > b$. The last step — solving the algebraic equation to get x as a function of t — is very hard. Find an equation expressing t as a function of x. Hint: substitute $u = \sqrt{b - x}$.

7. § A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere with radius 2 m and temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

Solve this *boundary value problem*, i.e. find a function $T(r)$ satisfying the above equation with $T(1)$ and $T(2)$ the prescribed amounts. Hint: first find the general form, by solving the first-order differential equation in $S = dT/dr$.

8. § The air in a room with volume $180\,\mathrm{m}^3$ contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of $2\,\mathrm{m}^3/\mathrm{min}$ and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

9. § When a raindrop falls, it increases in size and so its mass m at time t is a function of t: $m = m(t)$. The rate of growth of the mass if $km(t)$ for some positive constant k. Newton's Laws specify, moreover, that $(mv') = gm$, where $v = v(t)$ is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. Find an expression for the *terminal velocity* $\lim_{t\to\infty} v(t)$ in terms of g and k.

10. § Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \left([f(t)]^2 + [f'(t)]^2\right) dt$$

for all real x.

11. § Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.