Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Word Problems

1. § When a raindrop falls, it increases in size and so its mass m at time t is a function of t: $m = m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k. Newton's Laws specify, moreover, that $(mv)' = gm$, where $v = v(t)$ is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. Find an expression for the *terminal velocity* $\lim_{t\to\infty} v(t)$ in terms of g and k.

2. When an object moves through a fluid, the friction force on the object is generally a function of the velocity of the object. When the object moves slowly, a very good approximation is that the force is proportional to the velocity: $F = av$ for some constant a. When the object moves quickly, a very good approximation is that the force is proportional to the square of the velocity: $F = bv^2$. Recall that an object of mass m moving under a force F experiences an *acceleration* — change in velocity — following the formula $v' = F/m$.

   (a) Use physical intuition to explain why $a < 0$ for any value of v, and why $b < 0$ for positive v and $b > 0$ for negative v.

   (b) Marbles moving through honey and car shock absorbers moving through oil are good examples of the *viscous* approximation $F = av$. Assume that an object with mass m starts out moving with velocity v_0, slows down due to viscous friction, and has no other forces acting on it. Find its velocity as a function of time.

   (c) Baseballs flying through the air are good examples of the *turbulent* approximation $F = bv^2$. Assume that an object with mass m starts out moving with velocity v_0, slows down due to turbulent friction, and has no other forces acting on it. Find its velocity as a function of time.

   (d) Falling objects experience, in addition to whatever friction forces surround them, a constant downward acceleration g due to gravity. Find differential equations describing the velocity of an object of mass m moving under gravity and a (i) viscous, (ii) turbulent friction force.

   (e) Solve the two differential equations from part (d).

   (f) The *terminal velocity* of a falling object is its limit $\lim_{t\to\infty} v(t)$. Find the terminal velocity of an object falling in the (i) viscous and (ii) turbulent regimes.

3. § The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in at time t, the carrying capacity is estimated to be $K = 8 \times 10^7$ kg, and the relative growth rate is $k = 0.71$ per year.

(a) Based only on the type of differential equation, what can you say about the population of halibut?

(b) What are the equilibrium solutions? Which solutions are stable, and which are unstable? If the pacific fishery starts with some positive amount of halibut, and if the differential equation is perfectly satisfied, what will be the long-term population $y(\infty) = \lim_{t\to\infty} y(t)$? Hint: solve the problem in terms of the variables k and K, and only substitute in numbers at the very end.

(c) If the biomass of halibut starts at 2×10^7 kg, find the biomass a year later.

(d) If the biomass of halibut starts at 2×10^7 kg, how long will it take for the biomass to double?

4. (a) Consider the Pacific halibut fishery in the previous problem. If fishers harvest a biomass of L halibut per year, explain why

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right) - L$$

is a reasonable model for the population of halibut. For this problem, don't substitute numbers until the very end — work in terms of the unknown constants k, K, and L.

(b) What are the equilibrium populations? Are they stable or unstable? Explain qualitatively what will happen to the long-term fish population.

(c) Explain how your answers to the previous question depend on the size of L. In particular, what amount of harvesting is "too much", in the sense that there is no sustainable population of halibut if fishers harvest that much fish per year?

5. § Show that if P satisfies the logistic equation $P' = kP(1 - \frac{P}{K})$, then:

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

Deduce that a population grows fastest when it reaches half its carrying capacity.

6. § The *doomsday equation* for population growth is $y' = ky^{1+c}$ where k and c are positive constants. Solve the doomsday equation with initial condition $y = y_0$, where $y_0 > 0$. Show that there is a finite time $t = T$ (doomsday) such that $\lim\limits_{t\to T^-} y(t) = \infty$.

7. § Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \left([f(t)]^2 + [f'(t)]^2\right) dt$$

for all real x.

8. § Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a+b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.