Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Linear Differential Equations of First Order

An *operator* is a kind of function that takes *functions* as inputs and outputs *functions*. For example, the $\frac{d}{dx}$, which outputs the derivative of its input, is an *operator*; the multiplication by x is another operator. For now, we will be only interested in *differential operators*, which are composed of the basic operations and differentiation: $y \mapsto \frac{dy}{dx} + xy$ for example. Today we will focus on *first-order* differential operators, which do not involve second- and higher derivatives. An operator $\mathcal{F}$ is *linear* if it has the property that $\mathcal{F}[y_1(x) + y_2(x)] = \mathcal{F}[y_1(x)] + \mathcal{F}[y_2(x)]$ for any two functions y_1 and y_2. All the example operators we've listed so far are linear. A first-order linear differential operator is necessarily of the form

$$\mathcal{F}[y(x)] = a(x)\,\frac{dy}{dx} + b(x)\,y(x)$$

A *linear differential equation* is a differential equation of the form $\mathcal{F}[y(x)] = c(x)$, where $\mathcal{F}$ is a linear differential operator. By dividing by $a(x)$, and writing $P(x) = b(x)/a(x)$ and $Q(x) = c(x)/a(x)$, it suffices to consider linear differential equations of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Let's say that $\mathcal{F}$ is a linear operator, and that $y_1(x)$ and $y_2(x)$ are both solutions to the equation $\mathcal{F}[y(x)] = c(x)$. Since $\mathcal{F}$ is linear, we have:

$$\mathcal{F}[y_1(x) - y_2(x)] = \mathcal{F}[y_1(x)] - \mathcal{F}[y_2(x)] = c(x) - c(x) = 0$$

Conversely, if $y_1(x)$ is a solution to the equation $\mathcal{F}[y(x)] = c(x)$ and $y_0(x)$ is a solution to the equation $\mathcal{F}[y(x)] = 0$, then $y_1(x) + y_0(x)$ is a solution to $\mathcal{F}[y(x)] = c(x)$. The linear differential equation $\mathcal{F}[y(x)] = 0$ is called *homogeneous*, and the solutions to it are the *homogeneous solutions*. We have seen that the difference of any two solutions to $\mathcal{F}[y(x)] = c(x)$ is a homogeneous solution, and conversely, given any *particular solution* $y_p(x)$ to $\mathcal{F}[y(x)] = c(x)$, all other solutions are of the form $y_p(x) + y_0(x)$ where $y_0(x)$ ranges over the set of homogeneous solutions.

To make this more down-to-earth, let's consider a case you already understand. The operator $\frac{d}{dx}$ is a first-order linear differential operator. Its homogeneous differential equation is $\frac{d}{dx}[y] = 0$, and the solutions are $y = C$, where C is a constant. Given a function $c(x)$, the solutions to the inhomogeneous equation $\frac{d}{dx}[y(x)] = c(x)$ are given by $\int c(x)\,dx$; as you know, this set can be described by finding a single antiderivative of $c(x)$ and then adding "$+C$".

In any case, we now describe how to find solutions to all first-order linear differential equations. Recall that it suffices to consider equations of the form $y' + Py = Q$. The trick is to remember that the product rule turns a product into a sum: $(I(x)\,y(x))' = I(x)\,y'(x) + I'(x)\,y(x)$. So the idea is to find a function $I(x)$ so that $I(x)\,y'(x) + I'(x)\,y(x) = I(x)\,(y'(x) + P(x)\,y(x))$. I.e. we want a function $I(x)$ with $I'(x) = P(x)I(x)$. But this is a separable differential equation in $I(x)$: its solutions are of the form $I(x) = e^{\int P(x)dx}$. Thus, we multiply both sides of $y' + Py = Q$ by $I = e^{\int P}$, and then we get $(Iy)' = e^{\int I}Q$, or $Iy = \int IQ$, or

$$\boxed{y(x) = e^{-\int P(x)dx} \int \left(e^{\int P(x)dx} Q(x)\right) dx}$$

1. When $Q(x) = 0$, the linear differential equation $y'(x) + P(x)y(x) = 0$ is also a separable differential equation. Solve it as a separable equation, and also solve it using the above formula for linear differential equations.

2. To solve a linear differential equation, we had to choose an integrating factor $I(x) = e^{\int P(x)\,dx}$. But "$\int P(x)\,dx$" is defined only up to a constant. If you add a $+C$ to $\int P(x)\,dx$, what happens to the solution to the differential equation?

3. To solve a linear differential equation, we had to integrate $\int I(x)\,Q(x)\,dx$. What happens to the constant of integration here? How does this compare to your answer to problem 1.? Interpret the general solution to $y' + Py = Q$ as a "particular solution" plus a "homogeneous solution".

4. § Find the general solutions to the following differential equations:

   (a) $xy' + y = \sqrt{x}$ (b) $y' + y = \sin(e^x)$ (c) $\sin x \dfrac{dy}{dx} + (\cos x)y = \sin(x^2)$

   (d) $x\dfrac{dy}{dx} - 4y = x^4 e^x$ (e) $(1+t)\dfrac{du}{dt} + u = 1 + t$ (f) $t \ln t \dfrac{dr}{dt} + r = te^t$

5. § Solve the initial value problem:

   (a) $2xy' + y = 6x,\ x > 0,\ y(4) = 20$ (b) $xy' = y + x^2 \sin x,\ y(\pi) = 0$

6. § A tank contains 100 L of water. A solution with salt concentration 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

   $$\frac{dy}{dt} = 2\,\mathrm{kg/min} - \frac{(3\,\mathrm{L/min})\,y}{(100\,\mathrm{L}) + (2\,\mathrm{L/min})\,t}$$

   Solve this equation and find the concentration after 20 minutes.

7. § Use the *Bernoulli substitution* $u = y^{1-n}$ to turn the differential equation $y' + Py = Qy^n$ into a linear differential equation in u.

8. § Use the method in the previous problem to solve the following differential equations:

   (a) $xy' + y = -xy^2$ (b) $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$