Math 1B: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Summer1B/

Find two or three classmates and a few feet of chalkboard. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Complex Numbers

All the rules for manipulating complex numbers come from the definition: $i^2 = -1$. But some techniques are worth learning. To each complex number we associate its "complex conjugate", found by substituting $-i$ for i everywhere. This is a useful number, since the product of a number and its conjugate is always a positive real number. Another useful trick is the Euler formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1. Simplify the following expression:

$$\sqrt{\frac{5}{1-2i} + (2+i)(-1+i) + 2 - \sqrt{3}}$$

2. Find all solutions to the equation:

$$2x^2 - 2x + 1 = 0$$

3. The quadratic formula solves any equation of the form $ax^2 + bx + c = 0$ in complex numbers. In this problem, you'll derive a similar formula for cubic equations.

   (a) By substituting $z = w - 2/w$, solve the equation

   $$z^3 + 6z = 20$$

   (b) By substituting $z = w - p/(3w)$, find a formula for the three solutions to the equation

   $$z^3 + pz = q$$

   (c) By substituting $x = z + 2$ and using the formula in part (b), solve the equation

   $$z^3 - 6z^2 - 5z + 22 = 0$$

   (d) By substituting $x = z - a/3$ and using the formula in part (b), find a formula for the three solutions to the equation

   $$z^3 + az^2 + bz + c$$

4. Use Euler's formula to prove the following formulas for sine and cosine:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad\qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

5. Use Euler's formula to find "triple angle fomulas" expressing $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

6. (a) If $u(x) = f(x)+ig(x)$ is a complex-valued function of a real variable x, then the *derivative* of u is defined component-wise: $u'(x) = f'(x) + ig'(x)$. Let a and b be real constants. Use Euler's formula to evaluate

$$\frac{d}{dx}\left[e^{(a+ib)x}\right]$$

   (b) In the above set-up, the *indefinite integral* $\int u(x)\,dx$ of $u(x)$ is any antiderivative. Evaluate

$$\int e^{(1+i)x}\,dx$$

   (c) By considering real and imaginary parts of the above integral, compute

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$

   Or use the formulas from exercise 4.

7. Conversely, we can "prove" Euler's formula by demanding that the function $y = e^{ax}$ be solution to the differential equation $y' = ay$, even for complex values of a. In particular:

   (a) Let $y(x) = \cos x + i\sin x$. Find $y'(x)$.

   (b) Prove that $y'(x) = iy(x)$ for $y(x) = \cos x + i\sin x$.

   (c) Conclude that $y(x) = Ce^{ix}$ for some constant C.

   (d) What is $y(0)$? Use this initial value to find C, and hence prove Euler's formula.

8. In the previous exercise, we *defined* the function $y = e^{ax}$ by the property that $y' = ay$ (and the initial condition $y(0) = 1$). For this to be a good definition, it had better be true that $e^{ax}e^{bx} = e^{(a+b)x}$. Prove that if $y_1(x)$ satisfies $y' = ay$, and $y_2(x)$ satisfies $y' = by$, then the product $y_1(x)y_2(x)$ satisfies $y' = (a+b)y$.

9. In the previous exercise, you needed to use the product rule, but we haven't proved that the product rule holds for derivatives of complex-valued functions. Let $f_1(x) = u_1(x) + iv_1(x)$ and $f_2(x) = u_2(x) + iv_2(x)$, where u_1, u_2, v_1, v_2 are real-valued functions.

   (a) Express the product $f_1(x)f_2(x)$ as $u(x) + iv(x)$ for real-valued functions u and v; these should be expressed in terms of u_1, u_2, v_1, v_2.

   (b) Use the real-valued product rule to differentiate $u(x)+iv(x)$, in terms of the u_1, u_2, v_1, v_2 and their derivatives.

   (c) Conversely, express $f_1'(x)f_2(x)+f_1(x)f_2'(x)$ in terms of the u_1, u_2, v_1, v_2 and their derivatives.

   (d) Prove that your expressions from (b) and (c) are the same.

10. What did the logarithm of the primitive 16th root of unity say when all the pie was gone?