Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

$ay'' + by' + cy = 0$

Consider the differential equation

$$ay'' + by' + cy = 0$$

where a, b, and c are real numbers. Then $y = e^{rt}$ is a solution if and only if $ar^2 + br + c = 0$. If this equation has two distinct real roots, then the corresponding exponential functions are linearly independent, so their linear combinations exhaust all the solutions to the differential equation.

If $ar^2 + br + c = 0$ has only one root r, then e^{rt} is a solution to $ay'' + by' + cy = 0$, but so is te^{rt}.

If $ar^2 + br + c = 0$ has no real roots, then it must have two distinct complex roots $r = \alpha \pm i\beta$. By Euler's formula, $e^{(\alpha \pm i\beta)t} = e^{\alpha t}(\cos \beta t \pm i \sin \beta t)$, and so two linearly independent real solutions to the differential equation are $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$.

1. § Solve the following differential equations:

   (a) $y'' + 4y' + 4y = 0$ (b) $y'' - 8y' + 12y = 0$ (c) $8y'' + 12y' + 5y = 0$

2. § Solve the initial value problem:

   (a) $y'' - 2y' + 5y = 0$, $y(\pi) = 0$, $y'(\pi) = 2$ (b) $y'' + 12y' + 36y = 0$, $y(1) = 0$, $y'(1) = 1$

3. § Solve the boundary value problem, if possible:

   (a) $y'' - 6y' + 9y - 9$, $y(0) = 1$, $y(1) = 0$ (b) $9y'' - 18y' + 10y = 0$, $y(0) = 0$, $y(\pi) = 1$

4. (a) A frictionless spring is described by a second-order differential equation: the force (mass times acceleration) is proportional to the displacement of the spring. If the mass is m and the "spring constant" (constant of proportionality) is k, write and solve a differential equation to find the most general equation for the position of the spring. (You need to know whether k is positive or negative: draw a picture to figure out which direction the force should push.)

   (b) Often engineers place springs in viscous fluids in order to dampen the movement of the spring; if the fluid is jostled enough so that the internal flow is consistently turbulent, then the damping force will be proportional to the velocity (let's say with proportionality constant c). If $c^2 > 4mk$, what is the behavior of the spring?

(c) If $c^2 < 4mk$, then the solution to the differential equation is

$$\text{position} = e^{-ct/2m}\left(A\cos(t\omega) + B\sin(t\omega)\right)$$

By plugging into your differential equation, find the frequency ω.

Sketch the graph of this solution. Car shock absorbers are dampened springs; if your shock absorber has $c^2 > 4mk$, what would it feel like to go over a bump?

5. In this exercise you'll investigate how the second-order differential equation $ay'' + by' + cy = 0$ behaves when $a \to 0$.

   (a) What are the solutions to $by' + cy = 0$? What are the roots to $br + c = 0$?

   (b) If a is not zero, then there are two solutions to $ar^2 + br + c = 0$. Evaluate the limits as $a \to 0$, holding b and c constant, of these solutions. Hint: $\sqrt{1+x} \approx 1 + x/2$ when x is small.

6. If the characteristic equation has zero real roots, then the solution in general is of the form

$$e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$$

   If there are two real roots, we can use the same equation, but with the hyperbolic functions rather than the trigonometric ones. For what values of α and β is

$$y(x) = e^{\alpha x}\left(A\cosh(\beta x) + B\sinh(\beta x)\right)$$

   a two-parameter family of solutions to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

   if $b^2 > 4ac$?

7. Show that if r is a double root of some polynomial $p(x)$, then r is also a root of $p'(x)$. Hint: what does the fact that r is a double root tell you about the factorization of p?

8. Let r_1 and r_2 be the two roots of $ar^2 + br + c = 0$, where $b^2 > 4ac > 0$. Holding a and c constant, take the limit as $b \to \sqrt{4ac}$ of the solution

$$y = \frac{1}{r_1 - r_2}e^{r_1 t} + \frac{1}{r_2 - r_1}e^{r_2 t}$$

   to the differential equation $ay'' + by' + cy = 0$. Hint: L'Hôpital.

   Indicentally, the above y is the solution to the initial-value problem $ay'' + by' + cy = 0$, $y(0) = 0$, $y'(0) = 1$, provided that $ar^2 + br + c = 0$ has two real roots. Check your work by verifying that the limit is the solution to the corresponding initial value problem when $ar^2 + br + c = 0$ has exactly one root.

   If you are brave, repeat this problem by considering the case when $b^2 < 4ac$. Find the solution to the initial value problem $y(0) = 0$, $y'(0) = 1$, and find the limit as $b \to \sqrt{4ac}$ of this solution.