Math 1B: Discussion Exercises

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$ay'' + by' + cy = g(t)$

Recall that if y_1 and y_2 are solutions to the inhomogeneous equation $ay'' + by' + cy = g(t)$, then $y_1 - y_2$ is a solution to the homogeneous equation $ay'' + by' + cy = 0$. Therefore, the general solution to $ay'' + by' + cy = g(t)$ is:

$$y = y_p + C_1 y_1 + C_2 y_2$$

where y_1 and y_2 are linearly independent solutions to the homogeneous equation, and y_p is any particular solution to the inhomogeneous equation. Thus, we can solve inhomogeneous linear differential equations provided we have some methods to guessing particular solutions to them.

The method we will outline is called the method of "undetermined coefficients". It generalizes well to higher-order differential equations, but it does not work for all functions $g(t)$. The idea is as follows. Certain special functions — namely, polynomials, exponentials, sines and cosines, and sums and product of these — have the special property that upon repeated differentiation, the functions "cycle through" a finite list. For example, if we repeatedly differentiate $e^t \sin t$, we get $e^t(\sin t + \cos t)$, $e^t(2\cos t)$, $e^t(2\cos t - 2\sin t)$, etc., all of which are linear combinations of the functions $e^t \sin t$ and $e^t \cos t$. So by "cycle through a finite list" I mean that a function $f(t)$ is *special* in this sense if there is a finite set of functions $f_1(t), \dots, f_n(t)$ so that all derivatives of $f(t)$ (including f itself) are linear combinations of elements of the set.

In particular, let's say that $f(t)$ is a special function, and $\{f_1(t), \dots, f_n(t)\}$ the corresponding finite set of functions. Then $af''(t) + bf'(t) + cf(t)$ is definitely a linear combination of the same finite list of functions. Conversely, let's say that $g(t)$ is a special function and $\{g_1, \dots, g_n\}$ its finite list. Then for any constants $A_1, \dots, A_n$, the function $f(t) = A_1 g_1(t) + \cdots + A_n g_n(t)$ is special, and it's reasonable to hope that we find constants $A_1, \dots, A_n$ so that $af''(t) + bf'(t) + cf(t) = g(t)$, since finding such constants requires only that we solve a system of n linear equations in n unknowns.

Of course, not every system of n linear equations in n unknowns has a solution. It turns out that the only way we could fail to find a solution in the above paragraph is if there are some constants $B_1, \dots, B_n$ so that $B_1 g_1(t) + \cdots + B_n g_n(t)$ is a solution to the homogeneous equation $ay'' + by' + c = 0$. In this case, we have to expand our set $\{g_1, \dots, g_n\}$. We can use the fact, though, that $(tg)' = g + tg'$, and expand the set by multiplying each member by t. For a second-order linear differential equation, you may have to do this at most twice; for an nth-order differential equation, n times.

All in all, we get the following rules for guessing the form of particular solutions to differential equations:

- If $g(t) = e^{kt}p(t)$, where p is a polynomial of degree n, guess $y_p(t) = e^{kt}q(t)$, where q is a degree-n polynomial with undetermined coefficients.
- If $g(t) = e^{kt}p(t)\cos mt$ or $e^{kt}p(t)\cos mt$, guess $y_p(t) = e^{kt}q(t)\cos mt + e^{kt}r(t)\sin mt$.
- If $g(t) = g_1(t) + g_2(t)$, it might be simpler to solve each equation $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$ separately, and add the answers.
- If any $y(t)$ of the form of the guess $y_p(t)$ is itself a solution to the complementary equation $ay'' + by' + cy = 0$, you may have to multiply those terms by t or t^2.

1. § Write a trial solution for the following differential equations. Do not determine the coefficients.

   (a) $y'' + 9y' = 1 + xe^{9x}$ (b) $y'' + 3y' - 4y = (x^3 + x)e^x$

   (c) $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$ (d) $y'' + 4y = e^{3x} + x \sin 2x$

2. § Find the general solution for the following differential equations:

   (a) $y'' - 4y' + 5y = e^{-x}$ (b) $y'' + 2y' + y = xe^{-x}$

3. § Solve the initial value problem:

$$y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$$

4. Use the method of undetermined coefficients to find the general solution to the following first-order linear differential equation:

$$y' = e^{ax} \sin bx$$

   where a, b are constants. How would you normally solve this differential equation? Which method do you prefer?

5. Use the method of undetermined coefficients to find the general solution to the following first-order linear differential equation:

$$y' = x^n e^{ax}$$

   where a is a constant and n is a positive integer. How would you normally solve this differential equation? Which method do you prefer?

6. A spring with spring constant k, mass m, and damping constant c is hung vertically, so that it experiences a constant downward force mg, where g is the acceleration due to gravity. Find the *equilibrium solution*, i.e. find the position at which the spring will hang without moving. Then find the general solution. Explain how why for the purposes of solving problems with springs, we can ignore gravity if we measure the displacement from the equilibrium solution, rather than from the location in which the spring doesn't apply any force.

7. A series circuit contains a resistor with $R = 40\,\Omega$, an inductor with $L = 2\,\text{H}$, and a capacitor with $C = 0.0025\,\text{F}$. Let's assume that the initial charge on the capacitor is 0, and that the initial current is 0.

   (a) § If we add a battery to the circuit that applies a constant potential of $12\,\text{V}$, how will the circuit respond?

   (b) How will the circuit respond if we instead use a power source that applies an alternating potential of $12 \sin(50t/\text{s})\,\text{V}$?

8. Let f and g be special functions, so that all the derivatives of f are linear combinations of $\{f_1, \ldots, f_m\}$ and all derivatives of g are linear combinations of $\{g_1, \ldots, g_n\}$.

   (a) Prove that the sum $f + g$ is a special function by finding a finite list of functions so that all derivatives of $f + g$ are linear combinations of members of the list.

   (b) Prove that the product fg is a special function by finding a finite list of functions so that all derivatives of fg are linear combinations of members of the list. Hint: think about the product rule.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart.