Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Sequences

A *sequence* is an infinite list of numbers. In this class, and in most of math, we generally begin at 0: the beginning of a sequence is called the "zeroth" term, and then there's the "first" term, then the "second", etc. But really the indexing — is the sequence $\{a_0, a_1, a_2, \dots\}$ or $\{a_4, a_5, a_6, \dots\}$ — is largely irrelevant. In any case, the formal definition is that "A *sequence* is a function from the non-negative integers to the real numbers."

A sequence might or might not *converge* to a number, meaning it gets closer and closer to that number. A definition: $\lim_{n\to\infty} a_n = L$ if $|L - a_n|$ is small for sufficiently large n, i.e. if for any given $\epsilon > 0$ there exists an N such that for all $n > N$, $\epsilon > |L - a_n|$.

Whether, and to what, a sequence converges does not depend on the "early" values (or indeed on any given term) of a sequence. If a_n and b_n each converge, then so do $a_n + b_n$ and $a_n b_n$, and the limits add and multiply correctly. If $f(x)$ is a continuous function and a_n converges to some number a_∞, then $f(a_n)$ converges to $f(a_\infty)$.

If a sequence is monotonic — strictly increasing or strictly decreasing — and bounded, then it converges.

A sequence can also *converge to* $+\infty$ or to $-\infty$ (or perhaps it's better to say "diverge to $+\infty$"). a_n converges to $+\infty$ if the terms increase without bound, i.e. $\lim_{n\to\infty} a_n = +\infty$ if for any given $\epsilon > 0$ there exists an N such that for all $n > N$, $\epsilon < a_n$. An unbounded monotonic sequence converges to infinity.

1. § Show that the sequence defined by $a_0 = 2$ and $a_{n+1} = 1/(3 - a_n)$ is positive and decreasing. Hence it must be convergent. Find the limit.

2. (a) Show that the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$ converges. This takes two steps: show that each term is bigger than the previous, and show that no term is bigger than 3 (by induction: show that if a_n is less than 3, then so is a_{n+1}).

   (b) § Find the limit of the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$, by finding an equation the limit must satisfy.

3. § Determine whether the following sequences converge to a finite number, converge to $\pm\infty$,

or neither. Find the limits of the ones that converge.

(a) $a_n = \dfrac{n^3}{n+1}$ (b) $a_n = \dfrac{3^{n+2}}{5^n}$ (c) $a_n = \sqrt{\dfrac{n+1}{9n+1}}$

(d) $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^1 + 1}$ (e) $a_n = \cos(2/n)$ (f) $\{\arctan 2n\}$

(g) $\left\{\dfrac{\ln n}{\ln 2n}\right\}$ (h) $\{n \cos n\pi\}$ (i) $a_n = \ln(n+1) - \ln n$

4. (a) Let's say a sequence s_n diverges to $+\infty$. What is the limit of $1/s_n$ as $n \to \infty$? Justify your answer. How would your answer change if $s_n \to -\infty$?
   (b) Find a sequence t_n such that $t_n \neq 0$ for any n and $\lim_{n\to\infty} t_n = 0$, but such that t_n does not tend to $+\infty$ nor to $-\infty$.
   (c) If you know that $t_n > 0$ for every n and that $t_n \to 0$ as $n \to \infty$, then what can you say about $\lim_{n\to\infty} 1/t_n$?

5. § In this exercise, you will investigate the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$.
   (a) Show that if $0 \leq a < b$ then
   $$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n$$
   (b) Deduce that
   $$b^n\left[(n+1)a - nb\right] < a^{n+1}$$
   (c) Let $a = 1 + 1/(n+1)$ and $b = 1 + 1/n$ in part (b) to show that the the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing.
   (d) Let $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.
   (e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
   (f) Hence prove that the sequence a_n converges.

6. Consider the sequence of functions $f_n(x)$ defined by $f_n(x) = \left(1 + \dfrac{x}{n}\right)^n$. Thus, the sequence a_n in the previous problem is $a_n = f_n(1)$. A proof analogous to the one above shows that for any given x, $f_n(x)$ converges. You may assume that it does for this problem.
   (a) What is $f_n'(x)$? How does it relate to $f_n(x)$? Write a differential equation for f_n.
   (b) Let $f_\infty(x)$ by given by $f_\infty(x) = \lim_{n\to\infty} f_n(x)$. Let's assume that f_∞ is differentiable, and that it satisfies a differential equation corresponding to the $n \to \infty$ limit of the differential equation for f_n. What is this differential equation?
   (c) What is $f_\infty(0)$? Hence what is $f_\infty(x)$? Hence what is the limit of the sequence a_n in the previous exercise?

7. Consider the sequence $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ from the previous exercise. Use the binomial theorem to expand $f_n(x)$ as a polynomial. Describe the behavior of the coefficients of this polynomial as $n \to \infty$. For example, what happens to the coefficient of x^2?