Math 1B: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Summer1B/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Infinite Series

A *series* is an infinite sum of numbers. There are two sequences associated to each series. First of all, there's the sequence of terms in the sum (a.k.a. "summands"):

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \rightsquigarrow 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

Second, there's the sequence of partial sums:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \rightsquigarrow 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$$

We normally write the former explicitly:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

On the other hand, a series *converges* exactly if the sequence of partial sums converges, and the limit of the sequence of partial sums is the "value" of the series. Standard notation: the sequence of summands is a_n, the sequence of partial sums is $s_n = \sum_{k=0}^{n} a_k$, and if $\{s_n\}$ converges, then $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$.

Here are some facts about series:

- A series cannot converge unless the sequence of summands tends to 0. But the sequence of summands can go to 0 without the series converging.
- The *geometric series* $a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$. If it converges, then it converges to $a/(1-r)$. (A proof and generalization of this is in exercise 3.)
- If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n = AB$. This is called the "Cauchy product" or "discrete convolution" of the two series.
- This one isn't so much a fact as a technique. If we can write each a_n as a difference $a_n = b_n - b_{n+1}$, then $s_n = b_0 - b_{n+1}$, so the sum $\sum a_n$ converges if the sequence $\{b_n\}$ converges. Partial fractions are a good way to find such decompositions of terms as differences.

1. What is the sum
$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots ?$$

How about

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots ?$$

If k is some number strictly greater than 1, what is

$$\sum_{n=1}^{\infty} \left(\frac{1}{k}\right)^n = \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \frac{1}{k^4} + \ldots ?$$

2. § Determine whether the following series are convergent or divergent. If convergent, find the sums:

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ (b) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}^n}$ (c) $\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$

(d) $\sum_{n=0}^{\infty} \frac{1+3^n}{2^n}$ (e) $\sum_{n=0}^{\infty} [(.8)^n - (.3)^n]$ (f) $\sum_{n=0}^{\infty} (\cos 1)^n$

(g) $\sum_{n=0}^{\infty} \frac{3}{n(n+3)}$ (h) $\sum_{n=1}^{\infty} \arctan n$ (i) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

3. Remember how to evaluate the geometric series: if $|r| < 1$, then we can evaluate $S = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$ by mutiplying by r and subtracting:

$$\begin{array}{rrcl} & S & = & a + ar + ar^2 + ar^3 + \ldots \\ -\ (& r \cdot S & = & \qquad ar + ar^2 + ar^3 + \ldots \quad) \\ \hline & S - rS & = & a \\ \text{thus} & S & = & \dfrac{a}{1-r}. \end{array}$$

Use this method to compute the following sums:

(a) $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \frac{6}{243} + \ldots$ (b) $1 + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} + \frac{25}{32} + \frac{36}{64} + \ldots$

4. § When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on; this chain reaction is called the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money she or he receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.

(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.

(b) Show that $\lim_{n\to\infty} S_n = kD$, there $k = 1/s$ is the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

(c) In fact, the marginal propensities to save and to consume vary by socioeconomic class (among other things): poor people spend a larger proportion of the money they receive, and rich people save a larger proportion. If the government is trying to stimulate the economy, to whom should it give its money?