Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

The Harmonic Series

The *harmonic series* is the sum

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

The sum diverges, although it does so slowly. A classic proof that the harmonic series diverges is the fact that $\frac{1}{3}+\frac{1}{4} > \frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{2}$, etc., so the sum is at least $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots = 1+\frac{1}{2}\times\infty = \infty$.

The harmonic series can be factored:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right)$$

The first term is just $\sum_0^\infty \left(\frac{1}{2}\right)^n = 1/(1-\frac{1}{2}) = 2$. The second term can be factored again:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots = \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right)\left(1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots\right)$$

and repeated factorization gives:

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \dots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \dots\right)\dots$$
$$= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\dots = \prod_{\text{primes } p} \frac{p}{p-1}$$

Each term on the right-hand-side is summed using the geometric series. A *prime* is a positive number p with precisely two factors. If there were only finitely many primes, then the product on the right-hand-side would be a finite number. Thus, there must be infinitely many primes.

1. The following steps will give another proof that the harmonic series diverges:

   (a) Consider the *alternating harmonic series*:

   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$$

   Let A represent the alternating harmonic series and H the harmonic series. Explain why $A = H - H$.

(b) We will prove later that the alternating harmonic series converges, to some positive number. (In fact, $A = \ln 2$.) But if H were convergent, what must A converge to?

2. It's a fact, although rather difficult to prove, that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \frac{1}{36} + \dots$$

converges to $\pi^2/6$. (We will prove that it converges tomorrow.) Find a factorization of the above sum analogous to the factorization of the harmonic series. Use the fact that π^2 is irrational to prove that there are infinitely many prime numbers.

Series are a lot like integrals. If a_n be a sequence (starting at 0, say), let's define its *discrete integral* to be the sequence of partial sums, given by:

$$(Sa)_n = \sum_{k=0}^{n-1} a_k$$

for $n \geq 1$ and by $(Sa)_0 = 0$. Also, we define the *discrete derivative* of a_n to be the sequence:

$$(Da)_n = a_{n+1} - a_n$$

It's now straightforward to check the *discrete fundamental theorem of calculus*:

$$(DSa)_n = a_n \text{ and } (SDa)_n = a_n - a_0$$

Then the infinite series $\sum_{n=0}^{\infty} a_n$ is just $\lim_{n\to\infty}(Sa)_n$.

3. Compare the definition of the infinite series $\sum_{n=0}^{\infty} a_n$ to the definition of the improper integral $\int_0^{\infty} f(x)dx$.

4. Find a formula for the "discrete product rule": I.e. find a formula for $(D(ab))_n$, where a_n and b_n are sequences and the product sequence $(ab)_n$ is defined to be $(ab)_n = a_n b_n$.

5. Use your answer to the previous question to find a formula for the "discrete integration by parts".