Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

The Integral Test

Let $f(x)$ be a continuous function that is positive and decreasing, at least after some cut-off. Then $\int_0^\infty f(x)\,dx$ converges if and only if $\sum_0^\infty f(n)$ converges. Indeed, we have a much more precise theorem. Let $f(x)$ be a positive decreasing function on $[N,\infty)$, where N is an integer. Then:

$$\int_{N+1}^\infty f(x)\,dx \leq \sum_{n=N+1}^\infty f(n) \leq \int_N^\infty f(x)\,dx$$

If the infinite sum $\sum_1^\infty a_n$ converges to some number s, then we define the Nth *remainder* to be $\sum_{N+1}^\infty a_n = s - s_N$, where s_N is the Nth partial sum $\sum_1^N a_n$. The Nth remainder measures the error in estimating the infinite sum by the Nth partial sum. Thus, the integral test provides a bound on the error of the estimate.

A particularly important application of the integral test is to determine whether the p-series $\sum_1^\infty 1/n^p$ converges. Recall that the improper integral $\int_1^\infty 1/x^p\,dx$ converges if and only if $p > 1$; then the infinite sum follows the same rule.

1. § Determine whether the following series converge or diverge:

   (a) $\displaystyle\sum_1^\infty ne^{-n}$ (b) $\displaystyle\sum_1^\infty \frac{n+2}{n+1}$ (c) $\displaystyle\sum_1^\infty \frac{n^2}{n^3+1}$

   (d) $\displaystyle\sum_1^\infty \frac{3n+2}{n(n+1)}$ (e) $\displaystyle\sum_1^\infty \frac{1}{n^2-4n+5}$ (f) $\displaystyle\sum_2^\infty \frac{1}{n(\ln n)^2}$

   (g) $\displaystyle\sum_1^\infty \frac{e^{1/n}}{n^2}$ (h) $\displaystyle\sum_3^\infty \frac{n^2}{e^n}$ (i) $\displaystyle\sum_1^\infty \frac{n}{n^4+1}$

2. (a) For what values of p does $\sum 1/n^p$ converge?

   (b) For what values of p does $\sum 1/(n\,(\ln n)^p)$ converge? You may assume that the series starts after $n = 1$.

   (c) For what pairs of values (p_0, p_1) does

   $$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}}$$

   converge? You may assume that the series starts after $n = 1$.

(d) For what $(k+1)$-tuples $(p_0, p_1, \ldots, p_k)$ does

$$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}\,(\ln\ln n)^{p_2}\,\ldots\,(\underbrace{\ln\ldots\ln}_{k} n)^{p_k}}$$

converge? You may assume that the series starts late enough so as never to have 0s in the denominator.

3. § Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth partial sum of the harmonic series.

   (a) Draw a picture to prove that $\ln n \leq s_n \leq 1 + \ln n$.

   (b) Draw a picture to determine whether the sequence $\{s_n - \ln n\}$ is increasing, decreasing, or not monotonic.

   (c) Does the sequence $\{s_n - \ln n\}$ have a limit? How do you know?

4. § Find all values of c for which the following series converges:

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

5. It is a fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

   (a) Let's say you were to estimate the value of $\pi^2/6$ by summing the first ten terms of the above infinite series. How accurate is this estimate?

   (b) How many terms would you need to sum to calculate $\pi^2/6$ correct to ten decimal places?

6. The integral test provide both upper and lower bounds for the sizes of errors in estimating series. In this exercise, we will describe a better method for estimating series.

   (a) Let $f(x)$ be a positive decreasing function on $[1, \infty)$, and $a_n = f(n)$. Assume that $\sum_1^{\infty} a_n$ converges. Prove that the number

$$\left(\sum_{n=1}^{\infty} a_n \right) - \left(\sum_{n=1}^{N} a_n + \int_{N+1}^{\infty} f(x)\, dx \right)$$

   is positive.

   (b) By drawing a picture, show that the above number is less than $a_{N+1} = f(N+1)$.

   (c) Let's now assume that in addition to being decreasing and positive, $f(x)$ is also concave-up everywhere. Prove that:

$$\frac{1}{2} a_{N+1} \leq \left(\sum_{n=1}^{\infty} a_n \right) - \left(\sum_{n=1}^{N} a_n + \int_{N+1}^{\infty} f(x)\, dx \right) \leq a_{N+1}$$

7. Use the results from the previous exercise to estimate $\pi^2/6$ correct to ten decimal places.