Math 1B: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Summer1B/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

The Comparison Tests

The Comparison Test says the following: Let a_n and b_n be sequences with $0 \leq a_n \leq b_n$ for every n, or at least for every n after some cutoff. If $\sum_1^\infty b_n$ converges, then so does $\sum_1^\infty a_n$; conversely, if $\sum_1^\infty a_n$ diverges, then so does $\sum_1^\infty b_n$. The proof is straightforward, and the picture is almost the same as for the integral test. The positivity condition is very important. Indeed, if $a_n \leq b_n$, then $\sum_1^N a_n \leq \sum_1^N b_n$ for every N. If $b_n \geq 0$ and $\sum b_n$ converges, then we have $\sum_1^N a_n \leq \sum_1^\infty b_n$, and so the sequence of partial sums $\sum_1^N a_n$ is bounded. But bounded sequences do not have to converge. On the other hand, monotonic bounded sequences necessarily converge, and if all a_n are positive, then $\sum_1^N a_n$ is an increasing sequence.

The Limit Comparison Test is often easier to use in practice. Let a_n and b_n be positive sequences. If $\lim_{n\to\infty} a_n/b_n$ exists and is not zero, then $\sum a_n$ converges if and only if $\sum b_n$ converges. If $\lim a_n/b_n = 0$ and if $\sum b_n$ converges, then $\sum a_n$ converges (but not conversely). If $\lim a_n/b_n = \infty$ and if $\sum a_n$ converges, then $\sum b_n$ converges. I'll prove the first statement, leaving the rest to you. Let $\lim_{n\to\infty} a_n/b_n = L \neq 0$. Then eventually we have $L/2 \leq a_n/b_n \leq 2L$, so $(L/2)b_n \leq a_n \leq (2L)b_n$. Now let's assume that $\sum a_n$ converges; then by the comparison test $\sum (L/2)b_n$ converges, but then $\sum b_n$ must also converge. On the other hand, if $\sum b_n$ converges, then $\sum (2L)b_n$ converges, and so by the comparison test $\sum a_n$ converges.

1. (a) Let $a_n = \ln((n+1)/n)$. What is $\lim_{n\to\infty} a_n$?
   (b) By telescoping the sum, prove that $\sum_1^\infty a_n$ diverges.
   (c) Explain why a_n is a counterexample to the claim that if $\lim a_n = 0$, then $\sum a_n$ converges.
   (d) Use the fact that $\ln(1+x) \leq x$ to prove that the Harmonic Series diverges.

2. § Determine whether the following series converge or diverge:

   (a) $\displaystyle\sum_2^\infty \frac{n^3}{n^4-1}$ (b) $\displaystyle\sum_1^\infty \frac{n-1}{n^2\sqrt{n}}$ (c) $\displaystyle\sum_1^\infty \frac{4+3^n}{2^n}$

   (d) $\displaystyle\sum_1^\infty \frac{n^2-1}{3n^4+1}$ (e) $\displaystyle\sum_1^\infty \frac{1+\sin n}{10^n}$ (f) $\displaystyle\sum_2^\infty \frac{\sqrt{n}}{n-1}$

   (g) $\displaystyle\sum_1^\infty \frac{1}{\sqrt{n^3+1}}$ (h) $\displaystyle\sum_1^\infty \frac{1}{2n+3}$ (i) $\displaystyle\sum_1^\infty \frac{n+4^n}{n+6^n}$

   (j) $\displaystyle\sum_{n=3}^\infty \frac{n+2}{(n+1)^3}$ (k) $\displaystyle\sum_{n=1}^\infty \frac{n^2-5n}{n^3+n+1}$ (l) $\displaystyle\sum_{n=1}^\infty \frac{n+5}{\sqrt[3]{n^7+n^2}}$

3. Let $\sum a_n$ be a convergent series and $\{b_n\}$ be a convergent sequence, and let a_n and b_n be positive for every n. Show that $\sum a_n b_n$ converges.

4. State the limit comparison test in terms of divergence. I.e. fill in the blanks: "If ..., and if ... diverges, then ... diverges."

5. Prove the parts of the limit comparison test that were not proved above, namely:

   - Let a_n, b_n be positive sequences with $\lim_{n\to\infty} a_n/b_n = 0$. If $\sum b_n$ converges, then $\sum a_n$ converges.
   - Let a_n, b_n be positive sequences with $\lim_{n\to\infty} a_n/b_n = \infty$. If $\sum a_n$ converges, then $\sum b_n$ converges.

6. § Give an example of a pair of series $\sum a_n$ and $\sum b_n$ with positive terms where $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum b_n$ diverges, but $\sum a_n$ converges.

7. Give an example of a pair of series $\sum a_n$ and $\sum b_n$ with positive terms where $a_n \le b_n$ and $\sum b_n$ diverges, but $\sum a_n$ converges.

8. Give an example of a pair of series $\sum a_n$ and $\sum b_n$ where $a_n \le b_n$ and $\sum b_n$ converges, but $\sum a_n$ diverges. Hint: what condition in the comparison test have we left out?

9. (a) § If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin a_n$ is also convergent?

   (b) What extra condition do you need to make the following statement true: "If $\sum \sin a_n$ converges and a_n are positive, then $\sum a_n$ converges."?

10. Use the comparison test to prove that for any positive x, $\sum_{n=0}^{\infty} x^n/n!$ converges.

11. (a) For what values of p does $\sum 1/n^p$ converge?

    (b) For what values of p does $\sum 1/(n\,(\ln n)^p)$ converge? You may assume that the series starts after $n = 1$.

    (c) For what pairs of values (p_0, p_1) does

    $$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}}$$

    converge? You may assume that the series starts after $n = 1$.

    (d) For what $(k+1)$-tuples $(p_0, p_1, \ldots, p_k)$ does

    $$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}\,(\ln\ln n)^{p_2}\,\ldots\,(\underbrace{\ln\ldots\ln}_{k} n)^{p_k}}$$

    converge? You may assume that the series starts late enough so as never to have 0s in the denominator.

    (e) Let $p(r) = p_0 + p_1 r + p_2 r^2 + \cdots + p_k r^k$. Remember that for small r, the geometric series is $1/(1-r) = 1 + r + r^2 + r^3 + \ldots$. Prove that the sum in part (d) converges if and only if $p(r) < 1/(1-r)$ for all sufficiently small positive r, and diverges if and only if $p(r) > 1/(1-r)$ for all sufficiently small positive r.