Math 1B: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Summer1B/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Alternating Series and Absolute Convergence

Let b_n be a positive decreasing sequence: $b_n \geq b_{n+1} \geq 0$ for every n. Then $\sum(-1)^n b_n$ converges. Moreover, if we truncate the series after then Nth term, and estimate $\sum_0^\infty (-1)^n b_n$ by $s_N = \sum_0^N (-1)^n b_n$, then the error of the estimate is at most $|b_{N+1}|$.

A series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges. It is a theorem that if $\sum a_n$ is absolutely convergent, then it is convergent. But many series are convergent without being absolutely convergent. For example, the Alternating Harmonic series $\sum_1^\infty (-1)^{n-1}/n$ converges by the alternating series test, but $\sum_1^\infty \left|(-1)^{n-1}/n\right| = \sum_1^\infty 1/n$ is the divergent Harmonic series. A series that converges but does not absolutely converge is said to *converge conditionally*.

1. For what values of p does $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^p}$
   (a) converge absolutely?
   (b) converge conditionally?
   (c) diverge?

2. For what values of r does $\displaystyle\sum_{n=1}^\infty r^n$
   (a) converge absolutely?
   (b) converge conditionally?
   (c) diverge?

3. § Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

   (a) $\displaystyle\sum_1^\infty \frac{(-1)^n n}{n+2}$ (b) $\displaystyle\sum_2^\infty \frac{(-1)^n}{\sqrt{n}}$ (c) $\displaystyle\sum_1^\infty \frac{(-1)^{n-1}}{\ln(n+4)}$

   (d) $\displaystyle\sum_1^\infty (-1)^n \frac{n}{\sqrt{n^3+2}}$ (e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{10^n}$ (f) $\displaystyle\sum_1^\infty \frac{\cos \pi n}{n^2}$

   (g) $\displaystyle\sum_1^\infty (-1)^n (-1)^n \cos\frac{\pi}{n}$ (h) $\displaystyle\sum_1^\infty \left(-\frac{n}{5}\right)^n$ (i) $\displaystyle\sum_1^\infty \left(-\frac{1}{2n}\right)^n$

4. § How many terms of the series would you need to add in order to find the sum to the indicated accuracy?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$, $|\text{error}| < 0.000005$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$, $|\text{error}| < 0.0$

5. § Show that the series $\sum(-1)^{n-1} b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the alternating series test not apply?

6. (a) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ diverges, but $\sum_{n=1}^{\infty}(a_n)^2$ converges.
   (b) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty}(a_n)^2$ diverges.

7. The *Riemann ζ function* is defined to be the "analytic continuation of" $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.
   (a) For what s does the above definition of $\zeta(s)$ converge? I.e. what is the domain of the right-hand-side?
   (b) Prove that when both sides converge, we have:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - \frac{1}{2^{s-1}}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)$$

   For what s does the left-hand-side converge?
   (c) Use the above equation to write a formula for $\zeta(s)$ that extends the domain to $(0, 1) \cup (1, \infty)$.
   (d) When $s = 0$, explain why the LHS of the above equation is the geometric series $\sum_0^{\infty} r^n$ with $r = -1$. Assuming that $\lim_{s\to 0} \sum_1^{\infty}(-1)^{n-1}/n^s = \lim_{s\to 0} \sum_0^{\infty} r^n$, find $\zeta(0) = \lim_{s\to 0}\zeta(s)$.

8. Make sense of the following proof from *Proofs without Words: Exercises in Visual Thinking* by Roger B. Nelsen (1993):