Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Absolute Convergence

Consider a series $\sum a_n$. We distinguish three cases. First, $\sum a_n$ might diverge. The Divergence Test provides the easiest proof that a series diverges. Second, it's possible that $\sum a_n$ converges but that $\sum |a_n|$ does not, because the negative terms in $\sum a_n$ cancel the positive terms. In this case, the series $\sum a_n$ is said to be *conditionally convergent*. In the third case, $\sum |a_n|$ converges, and $\sum a_n$ is called *absolutely convergent*. Of course, if a_n is already a positive sequence, then $|a_n| = a_n$ and so we have only the two cases.

Why are there only these three cases? In particular, why can't we have a series such that $\sum |a_n|$ converges by $\sum a_n$ diverges? One proof goes like this: let $(a_n)_+ = a_n$ is a_n is positive and 0 if a_n is negative, and let $(a_n)_- = -a_n$ if a_n is negative and 0 if a_n is positive. I.e. $(a_n)_+ = \frac{1}{2}(a_n + |a_n|)$ and $(a_n)_- = \frac{1}{2}(|a_n| - a_n)$. Thus, $(a_n)_+$ and $(a_n)_-$ are each positive sequences with $a_n = (a_n)_+ - (a_n)_-$ and $|a_n| = (a_n)_+ + (a_n)_-$. In particular, $(a_n)_\pm$ are each less than or equal to $|a_n|$. If $\sum |a_n|$ converges, then each of $\sum (a_n)_\pm$ must converge by the comparison test. But the difference of convergent series converges, and $\sum a_n = \sum (a_n)_+ - \sum (a_n)_-$.

Riemann proved the following theorem. Assume that $\sum a_n$ converges absolutely, and that b_n is any rearrangement of the sequence a_n. Then $\sum b_n$ converges absolutely, to the same value as $\sum a_n$. The proof is in two parts. First, we prove the theorem for positive series, by appealing to the monotone sequence theorem. Then we use the fact that $\sum a_n = \sum (a_n)_+ - \sum (a_n)_- = \sum (b_n)_+ - \sum (b_n)_- = \sum b_n$, since $(b_n)_\pm$ is a rearrangement of $(a_n)_\pm$.

But it is the converse of Riemann's theorem that is amazing. Assume that $\sum a_n$ converges conditionally. Then for any number S, there is a rearrangement b_n of the sequence a_n such that $\sum b_n = S$. Again we only outline the proof. We saw that if $\sum (a_n)_+$ and $\sum (a_n)_-$ both converge, then $\sum a_n$ converges absolutely. So at least one of $\sum (a_n)_\pm$ must diverge. But if only one diverges, then $\sum a_n =$ (converge) $-$ (diverge) or the other way around, which diverges. So both $\sum (a_n)_+$ and $\sum (a_n)_-$ diverge since we assumed that $\sum a_n$ converged conditionally. This means that for any number M, and for any number J, there is a number K such that $M \leq \sum_{n=J}^{K} (a_n)_+$, and similarly for $(a_n)_-$. Thus, to make the sequence b_n, we begin by adding only the positive terms of a_n until the partial sum is more than S. Now we subtract off the negative terms until the partial sum is less than S. Add, subtract, etc. So if the sequence of partial sums converges at all, it must converge to S. But does it converge? Well, let's be trickier, and always stop adding positive or negative terms just when we pass S. Then the difference from the partial sum to S is never more than some a_k, and $\lim a_k = 0$ (since $\sum a_n$ converges), so the difference get closer and closer to S.

1. We've introduced the following tests so far: Divergence Test, Integral Test, Comparison Test, Limit Comparison Test, Alternating Series Test. By thinking about the conditions of each test, classify the tests based on what they can say about a series. For example, the integral test

only applies to positive series, and so cannot decide whether a series converges conditionally; it can decide whether a series converges absolutely or not.

2. Recall the alternating harmonic series $\sum_1^\infty (-1)^{n-1}/n$. The AST proves that it converges to some number A, and the error estimate prove that $A > 0$. We gave a picture proof last time that $A = \ln 2$. Does $\sum_1^\infty (-1)^{n-1}/n$ converge conditionally or absolutely?

   Write out the first ten or so terms of $\sum_1^\infty (-1)^{n-1}/n$. Below that, write out the first five or so terms of $\frac{1}{2}\sum_1^\infty (-1)^{n-1}/n$, spaced so that the nth term of $\frac{1}{2}\sum_1^\infty (-1)^{n-1}/n$ lies beneath the $2n$th term of $\sum_1^\infty (-1)^{n-1}/n$. Add in columns to get a new series. This new series must converge, since it is the sum of convergent series, to $A + \frac{1}{2}A = \frac{3}{2}A$. On the other hand, prove that the new series is a rearrangement of $\sum_1^\infty (-1)^{n-1}/n$.

We now describe and prove a generalization of the Alternating Series Test. We consider a series of the form $\sum a_n b_n$ satisfying the following conditions: (1) b_n is a positive decreasing sequence with $\lim b_n = 0$; (2) there is some number M so that all the partial sums $s_N = \sum_0^{N-1} a_n$ satisfy $|s_N| \leq M$. Then we will prove that the series $\sum a_n b_n$ converges. This generalizes the AST, where we have $a_n = (-1)^n$, and so s_N is either 1 or 0 and $M = 1$.

Let's consider the partial sum $\sum_0^{N-1} a_n b_n$. In the notation from last week, se could write this as $(S[ab])_N$; the sequence of partial sums is a "discrete integral" of the sequence $(ab)_n = a_n b_n$. If we really think of it as an integral, then it's reasonable to try to use some sort of "integration by parts" to compute it. Let c_n and d_n be sequences, and recall the notation $(Dc)_n = c_{n+1} - c_n$ for the discrete derivative. Then the *discrete product rule* says that $D(cd) = (Dc)d + c(Dd) + (Dc)(Dd)$. We can make this look a little bit more like the product rule, at the cost of shifting one of the sequences. Using the fact that $d_{n+1} = d_n + (Dd)_n$, we have $D(cd)_n = (Dc)_n d_{n+1} + c_n (Dd)_n$. Let's now let $a_n = (Dc)_n$ and $b_n = d_{n+1}$. If we also assume that $c_0 = 0$, then we have $c_n = Sa_n$. Then we have:

$$\begin{aligned}
D(cd)_n &= (Dc)_n d_{n+1} + c_n (Dd)_n \\
D((Sa)_n b_{n-1}) &= a_n b_n + (Sa)_n (Db)_{n-1} \\
a_n b_n &= D((Sa)_n b_{n-1}) - (Sa)_n (Db)_{n-1} \\
S(a_n b_n) &= SD((Sa)_n b_{n-1}) - S((Sa)_n (Db)_{n-1}) \\
&= (Sa)_n b_{n-1} - S((Sa)_n (Db)_{n-1})
\end{aligned}$$

We have used condensed notation. In the Σ notation, this says:

$$\sum_{n=0}^{N-1} a_n b_n = \left(\sum_{n=0}^{N-1} a_n \right) b_{N-1} - \sum_{n=0}^{N-1} \left(\left(\sum_{k=0}^{n-1} a_k \right) (b_n - b_{n-1}) \right)$$

We are interested in the limit as $N \to \infty$ of the LHS. But on the RHS, the first term is the limit of something at most M times something going to 0, so vanishes by the Squeeze Theorem. (This is where we use that $b_n \to 0$.) We will prove that the second term converges absolutely. Indeed, the absolutely value of each summand is a number at most M times $b_n - b_{n-1}$, which is positive since b is decreasing. Thus, we use the comparison test, comparing the second sum to $\sum_1^{N-1} M(b_n - b_{n-1}) = Mb_0 - Mb_{N-1} \to Mb_0$. (The $n = 0$th term vanishes.) Thus both sums on the RHS have finite limits, so the LHS converges.

3. Prove that if $g(x)$ is a positive decreasing function and $f(x)$ is a continuous function such that there is some M with $M \geq \left| \int_0^x f(t)dt \right|$ for every x, then $\int_0^\infty f(x)g(x)dx$ converges. Use this to prove that $\int_0^\infty \frac{\sin x}{x} dx$ converges.