Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Ratio and Root Tests

The Ratio and Root Tests have a very similar form. Let $\sum a_n$ be a series. The Ratio Test begins by asking you to compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, and the Root Test begins by asking you to computer $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. It is a theorem that if both these limits exist, then they are equal (for a given series, either one or both may not exist, in which case the corresponding test is inconclusive), so let's call the common limit L. Then if $L < 1$, the series $\sum a_n$ converges absolutely, and if $L > 1$, then the series diverges. If $L = 1$, the tests are inconclusive.

In general, when $L = 1$, it is because the series $\sum |a_n|$ is comparable to a P-series $\sum 1/n^p$ (although this is certainly not always the case). If so, then you can decide whether $\sum |a_n|$ converges or diverges, and hence whether $\sum a_n$ is absolutely convergent or not. Of course, if $\sum a_n$ is absolutely convergent, then you can stop testing; if it is not absolutely convergent, then it may still be conditionally convergent, and a test like the Alternating Series Test may prove that $\sum a_n$ converges.

1. § Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ (h) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ (i) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

2. § Prove that $\sum_{n=0}^{\infty} x^n/n!$ converges for any x. What does this say about $\lim_{n\to\infty} x^n/n!$?

3. (a) It is a fact that $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. Use this fact and the Ratio Test to determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges.

   (b) It turns out that for the above series, the limit in the Root Test also exists. What is the limit in the Root Test, and what does part (a) say about its value?

   (c) Find all numbers x such that the Ratio and Root Tests are inconclusive when applied to the series:

   $$\sum_{n=1}^{\infty} \frac{x^n\, n!}{n^n}$$

(d) Use part (b) to justify the following estimate:

$$n! \sim \frac{n^n}{e^n}$$

(e) The estimate in part (d) is the leading-order part of *Stirling's Formula*, which says:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

or, more precisely,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Use this estimate to determine whether the following series converge of diverge:

i. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n\, n!}{n^n}$ ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{e^n\, n!}$ iii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{e^n\, n!}$

For part iii., you may assume that $\{n^n/(n!e^n)\}$ is a monotonic sequence.

4. § For which positive integers k does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ converge?

5. § Let $\sum a_n$ be a series with positive terms, and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so that $\sum a_n$ converges by the Ratio Test. Let R_n be the remainder after n terms, i.e. $R_n = a_{n+1} + a_{n+2} + a_{n+2} + \dots$.

   (a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that $R_n \le \dfrac{a_{n+1}}{1 - r_{n+1}}$.

   (b) If $\{r_n\}$ is an increasing sequence, show that $R_n \le \dfrac{a_{n+1}}{1 - L}$

6. (a) Let's say that $\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$. Prove that $\lim_{n\to\infty} a_n \neq 0$, and thus prove one part of the Ratio Test. Hint: Let $r_n = |a_{n+1}/a_n|$; then there is some N so that for $n \ge N$, $r_n > (L+1)/2 = K > 1$. Use this fact to prove that for $n \ge N$, we have $|a_n| \ge K^n|a_N|$. Since $K > 1$, $\lim_{n\to\infty} K^n|a_N| = \infty$, and so $\lim |a_n| = \infty$.

   (b) Let's say that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$. Prove that $\lim_{n\to\infty} a_n \neq 0$, and thus prove one part of the Root Test.

   (c) How would you modify these proofs if $\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$?

7. In this exercise, you'll prove the other half of the Ratio Test.

   (a) Let $\sum a_n$ be a series with positive terms, and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, and let $K = (L+1)/2$. Prove that there is some number N such that for $n \ge N$, we have $r_n \le K$.

   (b) Conclude that for $n \ge N$, we have $a_n \le K^n|a_N|$.

   (c) Use the comparison test and a geometric series to prove that $\sum_{n=N}^{\infty} a_n$ converges. Conclude that $\sum_1^{\infty} a_n$ converges. Hint: $K < 1$ (you must prove this).

   (d) Now let $\sum a_n$ be a series with possibly negative terms. Explain why parts (a)–(c) prove that if $\lim |a_{n+1}/a_n| < 1$, then $\sum a_n$ converges absolutely.

8. Modify and repeat the previous exercise to prove the Root Test.