# Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

## Power Series and Intervals of Convergence

A power series $\sum c_n(x-a)^n$ has an *interval of convergence*, which are the numbers $x$ such that the power series converges. Everything inside the interval of convergence is absolute; everything outside diverges. The boundaries you don't know: for each boundary point, decide if it's absolute, conditional, or divergent.) The interval is necessarily centered at $a$: it has a *radius of convergence*.

The proof of this is straightforward. Let's assume that $\sum c_n(x-a)^n$ converges when $x = w$, and let $z$ be any number with $|z-a| < |w-a|$. Since $\sum c_n(w-a)^n$ converges, by the divergence test eventually $|c_n(w-a)^n| < 1$. But $|c_n(z-a)^n| = |c_n||z-a|^n = |c_n|\left(\frac{|z-a|}{|w-a|}\right)^n |w-a|^n < \left(\frac{|z-a|}{|w-a|}\right)^n$. But $\frac{|z-a|}{|w-a|} < 1$ by assumption, so $\sum \left(\frac{|z-a|}{|w-a|}\right)^n$ converges as a geometric series, so $\sum |c_n(z-a)^n|$ converges by the comparison test, so $\sum c_n(z-a)^n$ converges absolutely.

1. § For what $x$ do the following series (i) converge absolutely? (ii) converge conditionally? (iii) diverge?

   (a) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ (f) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n}(x+1)^n$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$ (i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\cdot 4\cdot \cdots \cdot (2n-2)\cdot(2n)}$

2. If $\sum c_n(-4)^n$ diverges, does it follow that $\sum c_n(-6)^n$ diverges? What about $\sum c_n 4^n$?

3. § Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

4. (a) Let $R$ be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. Prove that if the limit exists, then

   $$R^{-1} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

   (b) Let $R$ be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. Prove that if the limit exists, then

   $$R^{-1} = \lim_{n\to\infty} \sqrt[n]{|c_n|}$$

(c) Conclude that if the limits in both the Ratio Test and the Root Test exist when applied to the same series, then the two limits are equal.

5. (a) § Let $p$ and $q$ be real numbers with $p < q$. Find a power series whose interval of convergence is:

   i. $[p, q]$  ii. $[p, q)$  iii. $(p, q]$  iv. $(p, q)$

   (b) § Is it possible for a power series to the interval $[0, \infty)$ as its interval of convergence? What about $(0, \infty)$?

6. § Suppose that the radius of convergence of the power series $\sum c_n x^n$ is $R$. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

7. (a) § Let $k$ be a positive integer. What is the radius of convergence of the following series?

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

   (b) Use Stirling's Formula, that $n! \approx \sqrt{2\pi n}(n/e)^n$, to determine whether the above series converges at the boundary of its interval of convergence.

8. § A function $f$ is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \dots$$

   That is, the coefficients of $f(x) = \sum_{n=0}^{\infty} c_n x^n$ are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geq 0$. Find the interval of convergence of this series and find an explicit formula for $f$.

9. § If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.

10. (a) What is the interval of convergence for the following series?

$$f(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

   (b) What is the value of $f(x)$? There are two ways to do this problem (so pick one):

      i. Multiply $f(x)$ by $x$, and subtract $f(x) - xf(x)$. Do you recognize this power series? Evaluate $f(x) - xf(x)$ and use that to solve for $f(x)$.

      ii. Integrate $F(x) = \int_0^x f(t)\,dt$ term-by-term. Do you recognize this power series? Evaluate $F(x)$ and differentiate to get $f(x) = F'(x)$.