Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Power Series

1. (a) Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. Prove that if the limit exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

   (b) Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. Prove that if the limit exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|}$$

   (c) Conclude that if the limits in both the Ratio Test and the Root Test exist when applied to the same series, then the two limits are equal.

If a power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R > 0$, then the integral $C + \sum_{n=0}^{\infty} c_n x^{n+1}/(n+1) = C + \sum_{n=1}^{\infty} c_{n-1} x^n/n$ and the derivative $\sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ also have radius of convergence equal to R. Endpoints are subtler: if $\sum c_n x^n$ converges at an endpoint, then so does its integral, and if it diverges at an endpoint, then so does its derivative.

Remember that we have the following power series representation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

2. (a) What is the interval of convergence for the following series?

$$f(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

   (b) What is the value of $f(x)$? There are two ways to do this problem (so pick one):

      i. Multiply $f(x)$ by x, and subtract $f(x) - x f(x)$. Do you recognize this power series? Evaluate $f(x) - x f(x)$ and use that to solve for $f(x)$.

      ii. Integrate $F(x) = \int_0^x f(t)\,dt$ term-by-term. Do you recognize this power series? Evaluate $F(x)$ and differentiate to get $f(x) = F'(x)$.

3. (a) Write down the power series expansion of $f(x) = 1/(1+x)$.

(b) Integrate both sides of your equation from part (a) to get a power series expansion of $\ln(1+x)$. What is the radius of convergence? Does this function converge at the endpoints? Use this to write down a series for $\ln(2)$.

(c) What is the integral of $\ln(1+x)$ in terms of functions (not power series)? Now, use the power series from (b) to get a power series for this integral, by multiplying, not integrating. How does this compare to the power series you get if you integrate the series in (b)?

4. Integrate the power series for $1/(1+x^2)$. What's the radius of convergence of your series? Does it converge at the endpoints? What equation do you get when you substitute $x=1$? What about when $x = 1/\sqrt{3}$?

5. By differentiating, find a power series representation for the function:

(a) $\dfrac{1}{(1-x)^2}$ (b) $\dfrac{1}{(1-x)^3}$ (c) $\dfrac{1}{(1-x)^n}$

(d) $\dfrac{1}{(1+x^2)^2}$ (e) $\dfrac{1}{(1+x^2)^3}$ (f) $\dfrac{1}{(1+x^2)^n}$

6. § Find a power series representation for the function and determine the interval of convergence:

(a) $\dfrac{3}{1-x^4}$ (b) $\dfrac{2}{3-x}$ (c) $\dfrac{x^2}{x+10}$ (d) $\dfrac{x}{4x+1}$

(e) $\dfrac{x}{x^2+16}$ (f) $\ln\left(x^2+4\right)$ (g) $\ln\left(\dfrac{1+x}{1-x}\right)$ (h) $\arctan 2x$

7. § Show that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

is a solution to the differential equation $f'' + f = 0$. What is the value of $f(0)$? Hence what is the function $f(x)$? What is the radius of convergence of the above series?

8. Use partial fractions to express the function

$$\frac{5x+1}{x^2-3x+2}$$

as a sum of power series. What is the interval of convergence?

9. (a) What power series do you get if you differentiate the power series for $1/(1-x)$?

(b) What power series do you get if you differentiate again?

(c) How would you write $\sum_{n=0}^{\infty} n^2 x^n$ as a function?

(d) If $p(x)$ is any polynomial, use the ratio test to determine the radius of convergence of $\sum_{n=0}^{\infty} p(n) x^n$. Does this converge on the boundary?

(e) Come up with a method that you could use to write $\sum_{n=0}^{\infty} p(n) x^n$ as a function for any given polynomial $p(n)$. For example, what is $\sum_{n=0}^{\infty} (3n^2 - 4n + 1) x^n$?

10. What is the power series of $\ln(1-x)$? What about $x\ln(1-x)$? Convince yourself that any integral of $x\ln(1-x)$ is a polynomial times $\ln(1-x)$. What about if $\ln(1-x)$ is replaced by $\arctan(x)$? Compare with exercise 3.