Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Taylor Series

If $f(x)$ can be represented by a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then f is ∞-times differentiable, and the coefficients are given by

$$\boxed{c_n = \frac{f^{(n)}(0)}{n!}}$$

These are called the *Taylor coefficients for f at* 0, and the series $\sum_{n=0}^{\infty} f^{(n)}(0)\, x^n/n!$ is called the *Taylor series for f centered at* 0, or the *Maclaurin series for f*. By translating, we can also define the *Taylor series centered at a*: $\sum_{n=0}^{\infty} f^{(n)}(a)\,(x-a)^n/n!$.

Some infinitely-differentiable functions cannot be represented by power series. But: the Taylor series $\sum_{n=0}^{\infty} f^{(n)}(0)\, x^n/n!$ converges to $f(x)$ for $|x| < R$ if f satisfies that

$$\lim_{n\to\infty} \left(\frac{R^n}{n!} \sup_{|x|<R} \left| f^{(n)}(x) \right| \right) = 0$$

Most functions that are actually used — e.g. rational functions, exponential and trigonometric functions and their inverses, etc. — are equal to their Taylor series inside the radius of convergence.

Functions that equal their Taylor series expansion inside the interval of convergence are called *analytic*. Here are a few analytic functions and their Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k}}{(2k)!} \qquad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^n}{n} \qquad \arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{2k+1}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

The last equation is the *binomial theorem*. The symbol $\binom{k}{n}$ is defined as $(k)(k-1)\dots(k-n+1)/n!$, for any number k and integer n. (The numerator is a product of k numbers, just like the denominator.)

1. Find the Taylor series expansion centered at 0 for $1/(1-x)$, by computing derivatives. Is this what you expect?

2. Write out the Taylor series expansion centered at 0 of the following functions. You should use a combination of direct differentiation and manipulation of power series. For each function, determine the radius of convergence.

(a) $\sin 2x$	(b) xe^x	(c) $\sqrt{1+x}$	(d) $\cosh x$
(e) $\ln(5+x)$	(f) $1/(2-x)$	(g) $\sin(2x-\pi/4)$	(h) $(3+x)^{1/3}$

3. § Let $f(x)=e^{x^2}$. Prove that $f^{(2n)}(0)=(2n)!/n!$. What is $f^{(2n+1)}(0)$?

4. (a) Fill in the following outline to prove that

$$\lim_{x\to 0} p(1/x)e^{-1/x^2}=0$$

   for any polynomial $p(x)$:
   - $e^{-1/x^2}<e^{-1/x}$ for $0<x<1$;
   - $\lim_{x\to\infty} x^n e^{-x}=\lim_{x\to 0} nx^{n-1}e^{-x}$ by L'Hospital's Rule;
   - $\lim_{x\to\infty} x^n e^{-x}=0$ by induction;
   - $\lim_{x\to\infty} p(x)e^{-x}=0$ by the sum rule;
   - $\lim_{x\to 0} p(1/x)e^{-1/|x|}=0$ by a substitution;
   - $\lim_{x\to 0} p(1/x)e^{-1/x^2}=0$ by the Squeeze Theorem.

   (b) Let $f(x)=e^{-x^2}$. Prove that any derivative of $f(x)$ is of the form $p(1/x)e^{-x^2}$ for some polynomial p.

   (c) It is a fact that the derivative of any function does not have a removable discontinuity. Use this fact to show that $f^{(n)}(0)=0$ for every n.

   (d) What is the Taylor series expansion of $f(x)=e^{-1/x^2}$ centered at 0? What is the radius of convergence of this series? Why should this make you troubled?

5. § By using the Taylor series expansion for $\sin(x)$ only up to the cubic term, approximate the non-zero solution for

$$x^2=\sin(x)$$

6. § Prove that

$$\int_0^1 \frac{1+x^{30}}{1+x^{60}}dx=1+\frac{c}{31}$$

   where $0<c<1$.

7. Let f_n be the Fibonacci sequence, given by the rules $f_0=0$, $f_1=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$. Let $F(x)=\sum_{n=0}^{\infty} f_n x^n$. You may ignore questions of convergence for this exercise.

   (a) Explain why the equation

$$\sum_{n=2}^{\infty} f_n x^n=\sum_{n=2}^{\infty} f_{n-1}x^n+\sum_{n=2}^{\infty} f_{n-2}x^n$$

   follows from the definition of the Fibonacci sequence.

   (b) Rewrite each side of the above equation in terms of x and $F(x)$.

   (c) Use your answer from part (b) to find an elementary expression for $F(x)$.

   (d) Use partial fractions to find an explicit power series for $F(x)$.

   (e) Thus find an explicit formula for f_n, the nth Fibonacci number.