Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Exercises marked with an § are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Others are my own or are independently marked.

Taylor Series

Let's say that we have a function $f(x)$ with Taylor series $\sum_{k=0}^{\infty} f^{(k)}(0)x^k/k!$. Let's add up only the first n terms, and define

$$T_n(x) \stackrel{\text{def}}{=} f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

If x is inside the radius of convergence of the power series, then $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)x^k/k! = \lim_{n\to\infty} T_n(x)$. How fast does this limit converge? Let's define the difference between $f(x)$ and $T_n(x)$ to be the "error"

$$R_n(x) \stackrel{\text{def}}{=} f(x) - T_n(x)$$

The strong form of Taylor's theorem gives a bound on the size of this error. If $a > 0$ and $|x| < a$, then

$$|R_n(x)| \leq \sup_{|z|\leq a} \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}a^n \stackrel{\text{def}}{=} E_n \tag{1}$$

The word "sup" is short for "supremum", and means that we take the z with $|z| \leq a$ that makes the $(n+1)$th derivative largest.

1. Use equation (1) to estimate the size of the error $\left|\sin(x) - (x - x^3/6)\right|$ when $|x| \leq 1$.

2. What are the derivatives of $\sin(x)$? Use this to bound $\sup_{|z|\leq a} \left|f^{(n+1)}(z)\right|$. Hence, for what a can you be sure that $\left|\sin(x) - (x - x^3/6 + x^5/120)\right| \leq 0.001$ for all $x \leq a$? For what odd number n can you be sure that $\left|\sin(x) - (x - x^3/6 + \cdots \pm x^n/n!)\right| \leq 0.001$ for $x \leq 10$?

3. Use equation (1) to estimate the error in the approximation
$$e^3 \approx 1 + 3 + 3^2/2 + 3^3/6 + 3^4/4! + 3^5/5!$$

4. According to special relativity, the total energy E of an object with mass m and momentum p satisfies $E^2 = m^2c^4 + p^2c^2$. Assuming that $p \ll mc$, use Taylor Series to expand the function $E(p)$ up to order p^2.

5. The restoring force on a pendulum at angle θ from vertical is such that $\theta = \theta(t)$ satisfies the differential equation
$$\theta'' = -\frac{g}{l}\sin\theta$$
where g is the gravitation constant and l is the length of the pendulum.

(a) When $\theta \approx 0$, explain why $\theta'' = -\omega^2\theta$ is a reasonable approximation for the above differential equation, where $\omega = \sqrt{g/l}$. If $\theta_{\max}$ is the maximal angle that the pendulum reaches, estimate the maximal error in the approximation.

(b) By multiplying both sides of the above differential equation by $2\theta'$ and by using the chain rule, explain why $\theta(t)$ satisfies the differential equation

$$\theta' = \sqrt{C + \frac{2g}{l}\cos\theta}$$

for some constant C. This is a separable differential equation; after some work it implies that the period T of the pendulum satisfies:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{d\theta}{1 - k^2\sin^2\theta}$$

where $k = \sin(\frac{1}{2}\theta_{\max})$ is a constant.

(c) § Use the binomial theorem with $x = k^2\sin^2\theta$ to expand the integrand in Taylor series. Then use the fact that

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\frac{1}{(2n)(2n-2)(2n-4)\dots(2)(1)} = \frac{\pi}{2}\frac{1}{2^n n!}$$

to find a formula for T as an infinite series in terms of k.

The rest of today's problems are from yesterday's handout, in case you didn't get to them last time.

6. § Let $f(x) = e^{x^2}$. Prove that $f^{(2n)}(0) = (2n)!/n!$. What is $f^{(2n+1)}(0)$?

7. § By using the Taylor series expansion for $\sin(x)$ only up to the cubic term, approximate the non-zero solution for

$$x^2 = \sin(x)$$

8. § Prove that

$$\int_0^1 \frac{1+x^{30}}{1+x^{60}}dx = 1 + \frac{c}{31}$$

where $0 < c < 1$.

9. Let f_n be the Fibonacci sequence, given by the rules $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. You may ignore questions of convergence for this exercise.

(a) Explain why the equation

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1}x^n + \sum_{n=2}^{\infty} f_{n-2}x^n$$

follows from the definition of the Fibonacci sequence.

(b) Rewrite each side of the above equation in terms of x and $F(x)$.

(c) Use your answer from part (b) to find an elementary expression for $F(x)$.

(d) Use partial fractions to find an explicit power series for $F(x)$.

(e) Thus find an explicit formula for f_n, the nth Fibonacci number.