Math 1B: First Exam
Tuesday, 24 July 2009

Instructor: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Summer1B/

Name: ANSWERS

Problem Number	1	2	3	4	5	Total
Score						
Maximum	20	20	20	20	20	100

Please do not begin this test until 2:10 p.m. You may work on the exam until 4 p.m. Please do not leave during the last 15 minutes of the exam time.

You must always justify your answers: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. Please box your final answers.

Calculators are not allowed. Please sign the following honor code:

I, the student whose name and signature appear on this midterm, have completed the exam by myself, without any help during the exam from other people, or from sources other than my allowed one-page hand-written cheat sheet. Moreover, I have not provided any aid to other students in the class during the exam. I understand that cheating prevents me from learning and hurts other students by creating an atmosphere of distrust. I consider myself to be an honorable person, and I have not cheated on this exam in any way. I promise to take an active part in seeing to it that others also do not cheat.

Signature: ______________________________

1. (a) (15 pts) Find the centroid of the region bound by the curves $y = \sqrt{x}$ and $y = x^2$.

By symmetry, the centroid definitely lies on the line $y = x$. Nevertheless, we compute both coordinates $\bar{x}$ and $\bar{y}$, using the following formulae:

$$\bar{x} = \frac{1}{A}\int_0^1 x\left(\sqrt{x} - x^2\right)dx$$
$$\bar{y} = \frac{1}{A}\frac{1}{2}\int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right)dx$$

Here A is the area of the figure, equal to $A = \int_0^1 \left(\sqrt{x} - x^2\right)dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$. We have used the fact that the two curves intersect at $(0,0)$ and at $(1,1)$, and that $\int_0^1 x^n dx = 1/(n+1)$. Then:

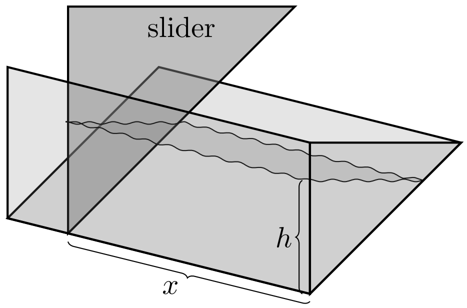
$$\bar{x} = \frac{1}{1/3}\int_0^1 \left(x^{3/2} - x^3\right)dx = 3\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{9}{20}$$
$$\bar{y} = \frac{1}{2/3}\int_0^1 \left(x - x^4\right) = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{9}{20}$$
$$(\bar{x}, \bar{y}) = \boxed{\left(\frac{9}{20}, \frac{9}{20}\right)}$$

(b) (5 pts) Recall the Theorem of Pappus, that the volume of the solid of revolution formed by rotating a region $\mathcal{R}$ around a line ℓ is the area of $\mathcal{R}$ times the distance that the centroid of $\mathcal{R}$ travels as it revolves around the line ℓ. Use this theorem and some geometry to find the volume of the solid of revolution formed by rotating the region bound by the curves $y = \sqrt{x}$ and $y = x^2$ around the line $y = -x$.

The shortest path from the centroid $(\frac{9}{20}, \frac{9}{20})$ to the line $y = -x$ is to the origin; the distance from the centroid to the origin is $\sqrt{(\frac{9}{20})^2 + (\frac{9}{20})^2} = \frac{9}{20}\sqrt{2}$. Thus, the distance traveled by the centroid is $\frac{9}{20}\sqrt{2} \times 2\pi = \frac{9}{10}\sqrt{2}\pi$, and so the volume is:

$$V = Ad = \frac{1}{3}\frac{9}{10}\sqrt{2}\pi = \boxed{\frac{3}{10}\sqrt{2}\pi}$$

2. A trough is constructed with vertical ends and flat sides that meet at a 45° point, so that each cross section is a right triangle, as shown in the picture. One end of the trough can move by sliding along the trough.



(a) (10 pts) If the trough is filled with water to a depth h, find the hydrostatic force on one end of the trough.

If the trough is filled to a depth h, then the submerged part of the end of the tough is a right $h \times h$ triangle. Thus, letting y denote the depth of a generic point, the width of the triangle at depth y is $h - y$, as y changes from 0 to h. So the hydrostatic force is:

$$F = \int_0^h \rho g y(h-y)dy = \rho g \left[h\frac{y^2}{2} - \frac{y^3}{3} \right]_0^h = \boxed{\rho g h^3/6}$$

Here $\rho = 1\,\mathrm{g/cm^3} = 1000\,\mathrm{kg/m^3}$ and $g = 10\,\mathrm{m/s^2} = 1000\,\mathrm{cm/s^2}$.

(b) (5 pts) Enough water is poured into the trough so that when the sliding end is such that the trough is one meter long, the water comes to a depth of ten centimeters. If you move the slider so that the trough has length x, how deep will the water now be?

The volume is a constant. The volume is equal to the length x times the area of an end. The end is an $h \times h$ triangle, so it has area $h^2/2$. Thus, the number $V = xh^2/2$ is a constant. Therefore, the depth h is given by

$$h = \sqrt{2V/x}$$

where $V = (1\,\mathrm{m})(0.1\,\mathrm{m})^2/2 = 0.005\,\mathrm{m}^3 = (100\,\mathrm{cm})(10\,\mathrm{cm})^2/2 = 5000\,\mathrm{cm}^3$. Since we know the volume, we can simplify a bit:

$$h = \sqrt{10,000\,\mathrm{cm}^3/x} = 100\,\mathrm{cm}\,(x/\mathrm{cm})^{-1/2} = \frac{1}{10}\,\mathrm{m}\,(x/\mathrm{m})^{-1/2}$$

(c) (5 pts) The *work* required to move the slider from a length a to a length b is defined to be $W = \int_b^a F(x)dx$, where $F(x)$ is the hydrostatic force on the slider when the slider is at length x. Find the work required to shrink the trough from a length of one meter to a length of fifty centimeters.

In part (a), we computed the force when the depth is h to be $F(h) = \rho g h^3/6$. In part (b), we computed $h = h(x) = \sqrt{2V/x}$. Thus, we have:

$$F(x) = \rho g \sqrt{2V/x}^3/6 = \frac{\rho g 2^{3/2} V^{3/2}}{6} x^{-3/2}$$

Then the work is:

$$\int_b^a \frac{\rho g 2^{3/2} V^{3/2}}{6} x^{-3/2} dx = \frac{\rho g 2^{3/2} V^{3/2}}{6} \left[\frac{1}{-1/2} x^{-1/2} \right]_b^a = \frac{\rho g 2^{3/2} V^{3/2}}{3} \left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}} \right)$$

where $V = 0.005\,\mathrm{m}^3 = 5000\,\mathrm{cm}^3$, $\rho g = 10,000\,\mathrm{kg/m^2 s^2} = 1000\,\mathrm{g/cm^2 s^2}$, and $a = 1\,\mathrm{m} = 100\,\mathrm{cm}$ and $b = 0.5\,\mathrm{m} = 50\,\mathrm{cm}$.

In terms of the numbers, we have:

$$W = \frac{\rho g}{3} (10000\,\mathrm{cm}^3)^{3/2} \left(\frac{1}{\sqrt{50\,\mathrm{cm}}} - \frac{1}{\sqrt{100\,\mathrm{cm}}} \right) = \frac{\sqrt{2}-1}{3} \times 10^8 \times \mathrm{g\,cm^2/s^2}$$

This is a much better number in mks units, whence:

$$\boxed{W = \frac{\sqrt{2}-1}{3}\,\mathrm{kg\,m^2/s^2} = 0.07\,\mathrm{kg\,m^2/s^2}}$$

where we have rounded $\sqrt{2}$ to 1.42 to make the numbers nice. This is about the accuracy of the estimate $g = 10\,\mathrm{m/s^2}$.

3. The class sizes at a certain large university are distributed exponentially: in a given semester there are $N(x) = Ae^{-x/B}$ classes of size x, where A and B are positive constants. Since universities are large, we approximate x and $N(x)$ (which actually can only take integer values) by continuous variables.

   (a) (5 pts) How many classes does the university offer in a given semester?

   We add up the total number of classes by integrating:

$$\text{total number} = \mathcal{N} = \int_0^\infty N(x)\,dx = \int_0^\infty Ae^{-x/B}dx = \boxed{AB}$$

   The integral starts at 0 because we cannot have a negative number of classes.

   (b) (5 pts) Write a probability distribution expressing the probability that a given class has size x.

   The probability distribution $\rho(x)$ should be proportional to $N(x)$ but normalized so that $\int_0^\infty \rho(x)dx$ is 1. Thus, we have

$$\rho(x) = \frac{1}{\mathcal{N}}N(x) = \boxed{\frac{1}{B}e^{-x/B}}$$

(c) (5 pts) What is the average class size at this university?

The average class size is:

$$\bar{x} = \frac{1}{\mathcal{N}} \int_0^\infty x\, N(x)\, dx = \int_0^\infty x\, \rho(x)\, dx = \frac{1}{B} \int_0^\infty x e^{-x/B}\, dx = \frac{1}{B} B^2 = \boxed{B}$$

(d) (5 pts) Jimmy Stewart is a student at this university. Since the probability that he ends up in any particular class is proportional to the number of students in that class, the probability that his first-period class has size x is proportional to $xAe^{-x/B}$. Find the expected size of Jimmy's first-period class.

From Jimmy's perspective, the probability of being in a class of size x is proportional to $M(x) = xe^{-x/B}$. Thus, the average class size is:

$$\bar{x} = \frac{\int_0^\infty x\, M(x)\, dx}{\int_0^\infty M(x)\, dx} = \frac{\int_0^\infty x^2\, e^{-x/B}\, dx}{\int_0^\infty x\, e^{-x/B}\, dx} = \frac{2B^3}{B^2} = \boxed{2B}$$

So Jimmy's average class is twice as large (!) as the number advertised by the university as the "average class size".

4. A certain population with constant carrying capacity K has a relative growth rate $k(t)$ that varies sinosoidally with time: $k(t) = a\sin(bt + c) + d$ for constants a, b, c, d.

   (a) (3 pts) Assuming that the relative growth rate is always positive, what can you say about the coefficients a, b, c, d?

   For $k(t)$ to always be positive, b and c can be arbitrary, but we must have $|a| < d$ (and in particular $d > 0$).

   (b) (2 pts) Write a differential equation modeling the population growth.

   We let $y(t)$ represent the population at time t, and we modify the logistic growth model:

$$\frac{dy}{dt} = k(t)\, y \left(1 - \frac{y}{K}\right) = \boxed{(a\sin(bt + c) + d)\, y \left(1 - \frac{y}{K}\right)}$$

(c) (15 pts) Assuming that the initial ($t = 0$) population is P_0, find a formula for the population at time t.

We solve the differential equation from part (b). It is separable, and we regret the choice to use d as a constant in the statement of the problem:

$$\begin{aligned} \frac{dy}{dt} &= (a\sin(bt+c)+d)\, y\left(1-\frac{y}{K}\right) \\ \int \frac{dy}{y\left(1-\frac{y}{K}\right)} &= \int (a\sin(bt+c)+d)\,dt \\ \ln\left|\frac{y}{y-K}\right| &= \frac{a}{b}\cos(bt+c)+dt+C \\ \frac{y}{y-K} &= Ae^{\frac{a}{b}\cos(bt+c)+dt} \end{aligned}$$

where C and $\pm A = e^C$ are unknown constants depending on the initial population. We have computed the integral by looking at a table of integrals, or by using partial fractions. When $t = 0$, we have $y = P_0$, and so:

$$e^{\cos c}A = \frac{P_0}{P_0 - K}$$

We will just write A for this number. We have:

$$\begin{aligned} y &= Ae^{\frac{a}{b}\cos(bt+c)+dt}\,(y-K) \\ \left(1-Ae^{\frac{a}{b}\cos(bt+c)+dt}\right) y &= -KAe^{\frac{a}{b}\cos(bt+c)+dt} \\ y &= \frac{KAe^{\frac{a}{b}\cos(bt+c)+dt}}{Ae^{\frac{a}{b}\cos(bt+c)+dt}-1} \\ &= \boxed{\frac{K}{1-A^{-1}e^{-\frac{a}{b}\cos(bt+c)-dt}}} \end{aligned}$$

5. (20 pts) A spring has mass $m = 1\,\mathrm{kg}$, damping constant $c = 2\,\mathrm{kg/s}$, and spring constant $k = 2\,\mathrm{kg/s^2}$, and the spring is driven by a force $F(t) = \sin(t/\mathrm{s})\,\mathrm{kg\,m/s^2}$, where t is the time. Find the general solution describing the displacement of spring as a function of time.

We let $y(t)$ represent the displacement at time t. Then $y(t)$ satisfies the differential equation

$$1\,\mathrm{kg}\,y'' + 2\,\mathrm{kg/s}\,y' + 2\,\mathrm{kg/s^2}\,y = \sin(t/\mathrm{s})\,\mathrm{kg\,m/s^2}$$

The auxiliary equation is

$$1\,\mathrm{kg}\,r^2 + 2\,\mathrm{kg/s}\,r + 2\,\mathrm{kg/s^2} = 0$$

and so $r = (-1 \pm i)/\mathrm{s}$. Thus, the homogeneous solutions are:

$$y_c = C_1 e^{-t/\mathrm{s}} \cos(t/\mathrm{s}) + C_2 e^{-t/\mathrm{s}} \sin(t/\mathrm{s})$$

We now look for a particular solution to the inhomogeneous equation. We guess that a particular solution is of the form

$$y_p = A\cos(t/\mathrm{s}) + B\sin(t/\mathrm{s})$$

whence

$$y_p' = -A\sin(t/\mathrm{s})\mathrm{s}^{-1} + B\cos(t/\mathrm{s})\mathrm{s}^{-1} \qquad y_p'' = -A\cos(t/\mathrm{s})\mathrm{s}^{-2} - B\sin(t/\mathrm{s})\mathrm{s}^{-2}$$

and so the left-hand-side of the differential equation becomes:

$$(-A\cos(t/\mathrm{s}) - B\sin(t/\mathrm{s}))\,\frac{\mathrm{kg}}{\mathrm{s}^2} + 2\,(B\cos(t/\mathrm{s}) - A\sin(t/\mathrm{s}))\,\frac{\mathrm{kg}}{\mathrm{s}^2} + 2\,(A\cos(t/\mathrm{s}) + B\sin(t/\mathrm{s}))\,\frac{\mathrm{kg}}{\mathrm{s}^2}$$

For this to equal $\sin(t/\mathrm{s})\,\mathrm{kg\,m/s^2}$, we must have:

$$-A + 2B + 2A = 0\,\mathrm{m} \qquad -B - 2A + 2B = 1\,\mathrm{m}$$

We solve this system of equations. We get:

$$A = -\frac{2}{5}\,\mathrm{m} \qquad B = \frac{1}{5}\,\mathrm{m}$$

Thus, the general solution to the inhomogeneous differential equation is

$$y_g = y_c + y_p = \boxed{C_1 e^{-t/\mathrm{s}} \cos(t/\mathrm{s}) + C_2 e^{-t/\mathrm{s}} \sin(t/\mathrm{s}) - \frac{2}{5}\cos(t/\mathrm{s}) + \frac{1}{5}\sin(t/\mathrm{s})}$$

References: All the problems on this midterm are due to the instructor, although they are loosely based on the material in *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. The honor-code language is adapted from the Stanford Honor Code (`http://www.stanford.edu/dept/vpsa/judicialaffairs/guiding/honorcode.htm`) and from the exams by Zvezda Stankova.

Feel free to use this page for extra scrap work.