Math 1B: Final Exam
Friday, 14 August 2009

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Name: ANSWERS

Problem Number	1	2	3	4	5	6	7	Total
Score								
Maximum	20	20	10	15	10	10	15	100

Please do not begin this test until 2:10 p.m. You may work on the exam until 4 p.m. Please do not leave during the last 15 minutes of the exam time.

You must always justify your answers: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. Please box your final answers.

Calculators are not allowed. Please sign the following honor code:

I, the student whose name and signature appear on this midterm, have completed the exam by myself, without any help during the exam from other people, or from sources other than my allowed one-page hand-written cheat sheet. Moreover, I have not provided any aid to other students in the class during the exam. I understand that cheating prevents me from learning and hurts other students by creating an atmosphere of distrust. I consider myself to be an honorable person, and I have not cheated on this exam in any way. I promise to take an active part in seeing to it that others also do not cheat.

Signature: ____________________________________

1. (20 pts total – 2 pts each) For each of the following statements, determine if the conclusion ALWAYS follows from the assumptions, if the conclusion is SOMETIMES true given the assumptions, or if the conclusion is NEVER true given the assumptions. You do not need to show any work or justify your answers for these question: only your answer will be graded.

   (a) (2 pts) If $\lim_{n\to\infty} a_n < 1$, then $\sum_1^\infty a_n$ converges.

   ALWAYS | **[SOMETIMES]** | NEVER

   (b) (2 pts) If the series $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ both converge, then $\sum_1^\infty (a_n + b_n)$ converges.

   [ALWAYS] | SOMETIMES | NEVER

   (c) (2 pts) If the sequences $\{a_n\}$ and $\{b_n\}$ both diverge, then $\{a_n b_n\}$ diverges.

   ALWAYS | **[SOMETIMES]** | NEVER

   (d) (2 pts) If $\sum_1^N a_n \geq -10$ for every N and if $a_n \leq 0$ for every n, then $\sum_1^\infty a_n$ converges.

   [ALWAYS] | SOMETIMES | NEVER

   (e) (2 pts) If $\sum_1^\infty c_n 2^n$ converges absolutely, then $\sum_1^\infty c_n(-2)^n$ converges conditionally.

   ALWAYS | SOMETIMES | **[NEVER]**

(f) (2 pts) If $\sum_1^\infty c_n(-2)^n$ converges conditionally, then $\sum_1^\infty c_n$ converges absolutely.

[ALWAYS] SOMETIMES NEVER

(g) (2 pts) If $\{b_n\}$ is a decreasing positive sequence, then $\sum_1^\infty (-1)^n b_n$ converges.

ALWAYS **[SOMETIMES]** NEVER

(h) (2 pts) If p is a real number, then the Ratio Test can be used to determine whether $\sum_1^\infty 1/n^p$ converges.

ALWAYS SOMETIMES **[NEVER]**

(i) (2 pts) If $0 \le a_n \le b_n$ and $\sum_1^\infty a_n$ converges, then $\sum_1^\infty b_n$ converges.

ALWAYS **[SOMETIMES]** NEVER

(j) (2 pts) If $\lim_{n\to\infty} b_n$ exists, then $\sum_1^\infty (b_n - b_{n+1})$ converges to b_1.

ALWAYS **[SOMETIMES]** NEVER

2. (20 pts total – 5 pts each) Determine whether each of the following series is ABSOLUTELY CONVERGENT, CONDITIONALLY CONVERGENT, or DIVERGENT. You must specify which test(s) you use for each series, and why the series satisfies the conditions of the test.

   (a) (5 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n!}$

   We use the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|-\frac{(n+2)/(n+1)!}{(n+1)/n!}\right|$$
$$= \lim_{n\to\infty}\frac{n+2}{(n+1)^2} = 0 < 1$$

   Therefore the series $\boxed{\text{converges absolutely}}$.

   (b) (5 pts) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n+1}$

   The series $\boxed{\text{diverges}}$ by the divergence test: $\lim_{n\to\infty} n^2/(n+1) = \infty$, and so the limit of the alternating sequence is not 0 (indeed, it does not exist).

(c) (5 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

The series satisfies the conditions of the Alternating Series Test, since $\ln n$ is an increasing positive function that tends to $+\infty$, and so $1/\ln n$ is descreasing, positive, and tends to 0. But the absolute sequence $\sum_2^\infty 1/\ln n$ diverges by, for example, comparison with $\sum 1/n$: $\ln n < n$ for every n, and so $1/\ln n > 1/n$. Thus the series $\boxed{\text{converges conditionally}}$.

(d) (5 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + \sqrt{n}}$

The series $\sum_{n=1}^\infty \frac{1}{n^3+\sqrt{n}}$ converges by Comparison or Limit Comparison with $\sum \frac{1}{n^3}$. Thus, the above series $\boxed{\text{converges absolutely}}$.

3. (10 pts) Find the interval of convergence of the following power series:

$$\sum_{n=2}^{\infty} \frac{(x-4)^n}{n \ln n \, 2^n}$$

We begin with the ratio test to determine the radius of convergence. We let $c_n = 1/(n \ln n \, 2^n)$. Then

$$\begin{aligned}
\text{ROC} &= \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| && \text{if this limit exists} \\
&= \lim_{n\to\infty} \left| \frac{(n+1)\ln(n+1)\, 2^{n+1}}{n \ln n \, 2^n} \right| \\
&= 2 \left(\lim_{n\to\infty} \frac{n+1}{n} \right) \left(\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \right) && \text{cancel and split the limits} \\
&= 2 \left(\lim_{n\to\infty} \frac{1}{1} \right) \left(\lim_{n\to\infty} \frac{1/(n+1)}{1/n} \right) && \text{L'Hospital} \\
&= 2 \lim_{n\to\infty} \frac{n}{n+1} = 2 && \text{evaluate and L'Hospital}
\end{aligned}$$

The interval of convergence is centered at 4, and so is either $(2,6)$, $(2,6]$, $[2,6)$, or $[2,6]$, depending on the convergence at the endpoints.

We now test these endpoints. At $x = 2$, the series is

$$\sum_{n=2}^{\infty} \frac{(2-4)^n}{n \ln n \, 2^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

This series satisfies the conditions of the alternating series test, as $n \ln n$ is the product of increasing positive functions and both tend to $+\infty$. So the power series converges at $x = 2$.

At $x = 6$, the series is

$$\sum_{n=2}^{\infty} \frac{(6-4)^n}{n \ln n \, 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

We use the integral test, since $1/(x \ln x)$ is a positive decreasing function. We have

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = \ln u \Big|_{\ln 2}^{\infty} = \infty$$

where we substituted $u = \ln x$. Since the integral diverges, the series does as well.

Thus, the interval of convergence of the power series is $\boxed{x \in [2,6)}$.

4. (15 pts) Evaluate the following definite integral as a series.

$$\int_0^1 \sqrt[3]{1+x^6}\,dx$$

You must both:

- Write your final answer in Σ notation, but you may leave your answer in terms of the binomial coefficients $\binom{k}{n}$.
- Write out the first four terms of the series.

We expand $\sqrt[3]{1+x^6}$ with the binomial theorem:

$$\sqrt[3]{1+x^6} = \sum_{n=0}^{\infty} \binom{1/3}{n} (x^6)^n = \sum_{n=0}^{\infty} \binom{1/3}{n} x^{6n}$$

$$\int_0^1 \sqrt[3]{1+x^6}\,dx = \int_0^1 \sum_{n=0}^{\infty} \binom{1/3}{n} x^{6n}\,dx$$

$$= \sum_{n=0}^{\infty} \binom{1/3}{n} \int_0^1 x^{6n}\,dx = \boxed{\sum_{n=0}^{\infty} \binom{1/3}{n} \frac{1}{6n+1}}$$

In the last line, we used the integral $\int_0^1 x^k\,dx = x^{k+1}/(k+1)\big|_0^1 = 1/(k+1)$ for $k \neq -1$.

We recall that

$$\binom{1/3}{n} = \frac{(\frac{1}{3})(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}$$

so that the numerator and denominator each has n terms in the product. Then the first four terms of the above sum are:

$$\sum_{n=0}^{\infty} \binom{1/3}{n} \frac{1}{6n+1} = \binom{1/3}{0}\frac{1}{0+1} + \binom{1/3}{1}\frac{1}{6+1} + \binom{1/3}{2}\frac{1}{12+1} + \binom{1/3}{3}\frac{1}{18+1} + \dots$$

$$= 1 + \frac{1}{3}\frac{1}{7} + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}\frac{1}{13} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{6}\frac{1}{19} + \dots$$

$$= \boxed{1 + \frac{1}{21} - \frac{1}{117} + \frac{5}{1539} + \dots}$$

5. (a) (5 pts) Find power series representations (centered at 0) for each of the following functions, and state the intervals of convergence for each series

- $\ln(1-x)$

  Quoting from a cheat sheet, or integrating $-1/(1-x)$, we have

  $$\boxed{\ln(1-x) = -\sum_{1}^{\infty} \frac{x^n}{n}, \quad x \in [-1,1)}$$

- $\ln(1-x^2)$

  We substitute x^2 for x in the above equation. Since x^2 is positive, if $x^2 \in [-1,1)$ then $x \in (-1,1)$, and:

  $$\boxed{\ln(1-x^2) = -\sum_{1}^{\infty} \frac{x^{2n}}{n}, \quad x \in (-1,1)}$$

- $\ln(1+x)$

  We substitute $-x$ for x in the above equation:

  $$\boxed{\ln(1+x) = \sum_{1}^{\infty} \frac{(-1)^{n-1}x^n}{n}, \quad x \in (-1,1]}$$

(b) (5 pts) Show that as power series, the representations for the above functions satisfy

$$\ln(1-x^2) = \ln(1-x) + \ln(1+x)$$

We work in the common domain $x \in (-1,1)$. The right-hand-side is:

$$-\sum_{1}^{\infty} \frac{x^{2n}}{n} + \sum_{1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = \sum_{1}^{\infty} \left(\frac{-1}{n} + \frac{(-1)^{n-1}}{n} \right) x^n$$

Then the terms cancel for odd n and add for even n, so that:

$$\text{RHS} = \sum_{n \text{ even}} \frac{-2}{n} x^n = \sum_{k=1}^{\infty} \frac{-2}{2k} x^{2k} = -\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$$

which is our series for $\ln(1-x^2)$.

6. (a) (2 pts) State the power series representation for $\arctan(x)$ centered at 0. What is its interval of convergence?

$$\boxed{\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad x \in [-1,1]}$$

(b) (4 pts) What is $\tan \pi/6$? Use this value for x in the power series representation to find a series that converges to $\pi/6$. Is the convergence absolute or conditional?

We have $\tan \pi/6 = \sqrt{3}/3$, and so $\arctan \sqrt{3}/3 = \pi/6$. Thus:

$$\boxed{\frac{\pi}{6} = \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{3}/3)^{2k+1}}{2k+1} = \frac{\sqrt{3}}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)3^k}}$$

The convergence is absolute because $\sqrt{3}/3$ is strictly within the interval of convergence.

(c) (4 pts) Use the error estimate for an Alternating Series to determine how many summands you would need in order to use this series to estimate $\pi/6$ correct to three decimal places ($|\text{error}| < 0.001$).

The Alternating Series test applies because $(2k+1)3^k$ is the product of increasing functions. The error estimate is that if we add up k terms, the error of the sum is at most $1/\big((2(k+1)+1)3^{k+1}\big)$. We would like this number to be less than 10^{-3}. Since $3^2 = 9 \approx 10$, we estimate $k+1 = 6$. Sure enough, then the error is at most $1/(13 \cdot 3^6) < 10^{-3}$ (as $3^6 > 100$). Indeed, $k+1 = 5$ gives an error of at most $1/(11 \cdot 3^5)$, and $3^5 = 243 > 100$, so even this is less than 10^{-3}. So $\boxed{k = 4}$ works.

7. (15 pts) Find a power-series representation (centered at 0) for the solution to the following initial value problem:

$$y'' - xy' - 2y = 0, \;\; y(0) = 1, \;\; y'(0) = 1$$

We assume that $y = \sum_0^\infty c_n x^n$ is a power series that satisfies the above differential equation. Then

$$xy' = x\sum_0^\infty nc_n x^{n-1} = \sum_0^\infty nc_n x^n$$

$$y'' = \sum_0^\infty n(n-1)c_n x^{n-2} = \sum_2^\infty n(n-1)c_n x^{n-2} = \sum_0^\infty (n+2)(n+1)c_{n+2}x^n$$

where in the second line we first recognized that the x^{-2} and x^{-1} terms were identically 0, and then reindexed $n \mapsto n+2$. Thus the differential equation reads:

$$0 = \sum_0^\infty (n+2)(n+1)c_{n+2}x^n - \sum_0^\infty nc_n x^n + \sum_0^\infty c_n x^n = \sum_0^\infty \left((n+2)(n+1)c_{n+2} - nc_n - 2c_n\right)x^n$$

This equation must hold for every x in some neighborhood of 0. Thus we must have:

$$0 = (n+2)(n+1)c_{n+2} - nc_n - 2c_n = (n+2)\left((n+1)c_{n+2} - c_n\right), \;\; n \geq 0$$

and so $c_n = c_{n-2}/(n-1)$ for $n \geq 2$. We also have the initial conditions $y(0) = 1 = y'(0)$. In terms of power series, Taylor's formula says that $y(0) = c_0$ and $y'(0) = c_1$. Thus:

$$c_0 = 1 \qquad\qquad c_1 = 1$$

$$c_2 = \frac{c_0}{1} = 1 \qquad\qquad c_3 = \frac{c_1}{2} = \frac{1}{2}$$

$$c_4 = \frac{c_2}{3} = \frac{1}{3} \qquad\qquad c_5 = \frac{c_3}{4} = \frac{1}{2\cdot 4}$$

$$c_6 = \frac{c_4}{5} = \frac{1}{3\cdot 5} \qquad\qquad \dots$$

The pattern is clear. We have:

$$\boxed{y = \sum_0^\infty c_n x^n \text{ where } c_n = \begin{cases} \dfrac{1}{1\cdot 3\cdot \dots \cdot (n-1)}, & n \text{ even} \\ \dfrac{1}{2\cdot 4\cdot \dots \cdot (n-1)}, & n \text{ odd} \end{cases}}$$

If we so desire, we can write this as follows:

$$y = \sum_{k=0}^\infty \frac{x^{2k+1}}{2^k k!} + \sum_{k=0}^\infty \frac{2^k k!}{(2k)!}x^{2k}$$

We conclude with the observation that this power series converges for every x.

References: All the problems on this midterm are due to the instructor, although they are loosely based on the material in *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. The honor-code language is adapted from the Stanford Honor Code (`http://www.stanford.edu/dept/vpsa/judicialaffairs/guiding/honorcode.htm`) and from the exams by Zvezda Stankova.

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