Math 1B: Quiz 2 — ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (5 pts) Evaluate the integral $\displaystyle\int \frac{x-1}{x^2+2x}\,dx$.

   We decompose the integrand into partial fractions:

$$\begin{aligned}
x^2+2x &= x(x+2) && \text{factoring}\\
\frac{x-1}{x^2+2x} &= \frac{A}{x}+\frac{B}{x+2} && \text{partial fraction ansatz}\\
x-1 &= A(x+2)+Bx && \text{cross-multiply}\\
0-1 &= A(0+2)+B0 && \text{when } x=0\\
-2-1 &= A(-2+2)+B(-2) && \text{when } x=-2\\
\frac{x-1}{x^2+2x} &= \frac{-1/2}{x}+\frac{3/2}{x+2} && \text{solving for } A \text{ and } B \text{ and substituting}
\end{aligned}$$

   Thus, we can evaluate the integral, using the fact that $\int \frac{1}{x+a}\,dx = \ln|x+a|+C$:

$$\begin{aligned}
\int \frac{x-1}{x^2+2x}\,dx &= \int\left(\frac{-1/2}{x}+\frac{3/2}{x+2}\right)dx\\
&= \boxed{-\frac{1}{2}\ln|x|+\frac{3}{2}\ln|x+2|+C}\\
&= \ln\sqrt{\left|\frac{(x+2)^3}{x}\right|}+C
\end{aligned}$$

   As with $\int dx/x$, the value of the constant C may change at any point of discontinuity; in this case, the value of C may change at $x=-2$ and at $x=0$.

2. (5 pts) Evaluate the integral $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$.

   We substitute $x = 2\sin\theta$, restricting ourselves to $-\pi/2<\theta<\pi/2$ (since $|x|\geq 2$ is outside the domain of the integrand). Then $dx = 2\cos\theta\,d\theta$ and:

$$\begin{aligned}
\int \frac{x^2}{(4-x^2)^{3/2}}\,dx &= \int \frac{(2\sin\theta)^2}{(4-4\sin^2\theta)^{3/2}}\,2\cos\theta\,d\theta\\
&= \int \frac{8\sin^2\theta\cos\theta}{8\cos^3\theta}\,d\theta\\
&= \int \tan^2\theta\,d\theta\\
&= \int\left(\sec^2\theta-1\right)d\theta\\
&= \tan\theta-\theta+C\\
&= \boxed{\frac{x}{\sqrt{4-x^2}}-\arcsin\frac{x}{2}+C}
\end{aligned}$$