Math 1B: Quiz 3 ANSWERS

GSI: Theo Johnson-Freyd Thursday, 2 July 2009

You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) You know that $\int_0^{2\pi} \sin^2 x\,dx = \pi$. But perhaps you don't know the numerical value of π — maybe you only know that $3 \leq \pi \leq 4$. If you were to use the midpoint rule to approximate $\int_0^{2\pi} \sin^2 x\,dx$, how many intervals would you need to take to assure that the error of your estimation is less than $0.00001 = 10^{-5}$?

   We recall that the error for the midpoint rule with n subintervals is at most

   $$E_M(n) \leq K\frac{(b-a)^3}{24n^2} = K\frac{(2\pi)^3}{24n^2}$$

   where $K \geq |f''(x)|$ for $f(x) = \sin^2 x$. We find the second derivative of $\sin^2$:

   $$(\sin^2 x)'' = (2\sin x \cos x)' = (\sin 2x)' = 2\cos 2x$$

   using the chain rule twice and trig identity. But $|\cos 2x|$ is never more than 1, so $K = 1$ works. Then

   $$E_M(n) \leq 1\frac{8\pi^3}{24n^2} \leq \frac{4^3}{3n^2} = \frac{64}{3}n^{-2}$$

   using $\pi \leq 4$.

   We have proved that for any n, the error with n intervals is less than $\frac{64}{3}n^{-2}$. We want to make sure that the error is less than 10^{-5}, so we look for an n so that $\frac{64}{3}n^{-2} \leq 10^{-5}$, i.e.:

   $$n^2 \geq \frac{64}{3} \times 10^5$$

   We round up $\frac{64}{3}$ to 22. Then we can take

   $$n = \sqrt{22 \times 10^5} = \sqrt{2.2 \times 10^6} = \sqrt{2.2} \times 10^3$$

   We round $\sqrt{2.2}$ up to 1.5, and so $\boxed{n = 1.5 \times 10^3 = 1500}$ works. Any answer less than about 3000, if correctly justified, is acceptable.

2. (0 pts) What are your plans for the long weekend?