Math 1B: Quiz 5 ANSWERS

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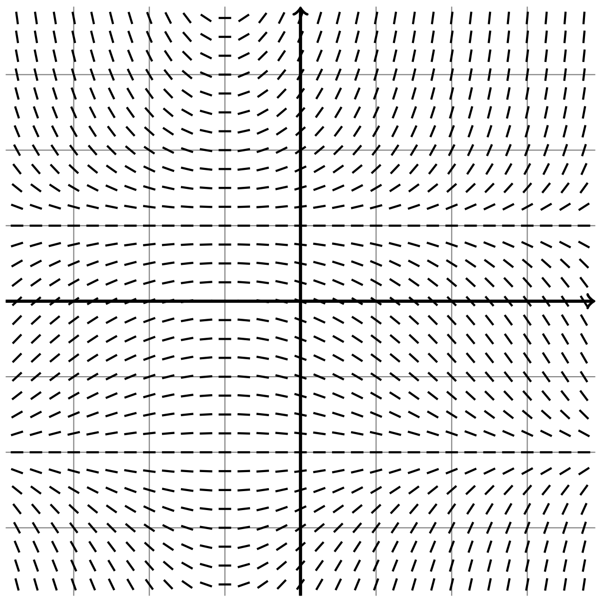
You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.



1. (2 pts) The above direction field is a graph of one of the following differential equations. Determine which it is, and explain your reasoning.

$$\frac{dy}{dx} = (y^2 - 1)\sin x \qquad \frac{dy}{dx} = \frac{1}{6}(x+1)(y-1)(y+2) \qquad \frac{dy}{dx} = \frac{2x}{3y}$$

The first differential equation has $y' = 0$ for $y = \pm 1$ and $x = k\pi$ for integers k. So it's out. The last has $y' = 0$ at $x = 0$ and $y' = \infty$ and $y = 0$, so it's out. The middle matches the graph.

2. (3 pts) What are the constant/equilibrium solutions to the differential equation graphed in the above direction field? Determine whether the equilibrium solutions are stable or unstable as $x \to +\infty$.

The equilibria solutions are the solutions $y = y(x)$ with $y' = 0$. We have $y' = 0$ for $x = -1$ and for $y = 1$ and $y = -2$; the first of these does not define a function. Thus, the equilibrium solutions are $y = 1$ and $y = -2$ — they are precisely the horizontal lines in the above direction field. $y = 1$ is unstable as $x \to +\infty$, since the field lines move away from it, whereas $y = -2$ is stable.

3. (5 pts) Find an equation for the solution to the differential equation graphed on the previous page that passes through the point $(1, -1)$. Sketch this solution on the previous graph. You do not need to simplify your answer.

We are considering the differential equation $\frac{dy}{dx} = \frac{1}{6}(x+1)(y-1)(y+2)$. It is separable; we solve it directly:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{6}(x+1)(y-1)(y+2) \\ \frac{6\,dy}{(y-1)(y+2)} &= (x+1)dx \\ \int \left(\frac{2}{y-1} - \frac{2}{y+2}\right) dy &= \int (x+1)dx \\ 2\ln|y-1| - 2\ln|y+2| &= \frac{x^2}{2} + x + C \end{aligned}$$

We pause here to find the value of C. When $x = 1$ we have $y = -1$, so:

$$2\ln|-2| - 2\ln|1| = \frac{1}{2} + 1 + C$$

or $\boxed{C = 2\ln 2 - \frac{3}{2}}$. Thus, we have:

$$\boxed{2\ln\left|\frac{y-1}{y+2}\right| = \frac{x^2}{2} + x + 2\ln 2 - \frac{3}{2}}$$

We don't need to for the quiz, but we can exponentiate and solve for y:

$$\frac{1-y}{y+2} = 2e^{-3/4}e^{x^2/4+x/2} \;\Rightarrow\; \boxed{y = \frac{1-4e^{-3/4}e^{x^2/4+x/2}}{1+2e^{-3/4}e^{x^2/4+x/2}}}$$