Math 1B: Quiz 6

ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) A certain population has no carrying capacity, and has a relative growth rate k. Humans harvest the population at a rate of $\ell\,(1+\cos(2\pi t/\mathrm{yr}))$, where ℓ is a constant and t is the time, measured in years. Let $P(t)$ be the population at time t.
   (a) Write a differential equation for $P(t)$ modeling the above description. Hint: you should get a linear differential equation.
   (b) Find the general solution to your differential equation.
2. (bonus) Is there necessarily a sustainable solution, in which the population never goes to 0 nor to ∞? Discuss the limitations of the model above. For example, is it reasonable to say that a population "has no carrying capacity"? Conversely, can humans harvest a fixed amount in all circumstances?

(a) The problem states that $P(t)$ grows with relative growth rate k and shrinks because of human harvesting. Thus, we have:

$$\frac{dP}{dt} = \left[\frac{dP}{dt}\right]_{\text{growth}} + \left[\frac{dP}{dt}\right]_{\text{harvest}} = kP - \ell\,(1+\cos(2\pi t/\mathrm{yr}))$$

Upon subtracting kP from both sides, we get a linear differential equation:

$$\boxed{\frac{dP}{dt} - kP = -\ell\,(1+\cos(2\pi t/\mathrm{yr}))}$$

(b) We solve the differential equation using the method of integrating factors. We multiply both sides by $I(t) = e^{-kt}$, and conclude that:

$$\frac{d}{dt}\left(e^{-kt}P\right) = e^{-kt}\frac{dP}{dt} - ke^{-kt}P = -\ell e^{-kt}\,(1+\cos(2\pi t/\mathrm{yr})) = -\ell e^{-kt} - \ell e^{-kt}\cos(2\pi t/\mathrm{yr})$$

We now integrate both sides with respect to dt:

$$e^{-kt}P(t) = -\int e^{-kt}dt - \ell\int e^{-kt}\cos(2\pi t/\mathrm{yr})dt$$
$$= \frac{1}{k}e^{-kt} - \ell e^{-kt}\frac{\frac{2\pi}{\mathrm{yr}}\sin(2\pi t/\mathrm{yr}) - k\cos(2\pi t/\mathrm{yr})}{(\frac{2\pi}{\mathrm{yr}})^2 + k^2} + C$$

The second integral was computed by parts. Thus:

$$\boxed{P(t) = \frac{1}{k} + \frac{\ell}{(\frac{2\pi}{\mathrm{yr}})^2 + k^2}\left(\frac{2\pi}{\mathrm{yr}}\sin(2\pi t/\mathrm{yr}) - k\cos(2\pi t/\mathrm{yr})\right) + Ce^{kt}}$$

(bonus) The sustainable solution would be when $C = 0$; it corresponds to an initial population $P(0) = \frac{1}{k} - \frac{kl}{(\frac{2\pi}{\mathrm{yr}})^2+k^2}$. This is reasonable only if this number is positive, i.e. if $l < 1 + (\frac{2\pi}{\mathrm{yr}})^2/k^2$. Since every population has a carrying capacity, the model is only accurate for P much smaller than the capacity. Also, the model is only accurate in situations in which the population does not go to 0, since then the humans cannot continue to harvest.