Math 1B: Quiz 6 ANSWERS

GSI: Theo Johnson-Freyd — Friday, 24 July 2009

You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) A bob of mass $m = 1\,\mathrm{kg}$ is on the end of a spring with spring constant $k = 3\,\mathrm{kg/s^2}$, and placed in a viscous fluid providing a frictive damping constant $c = 4\,\mathrm{kg/s}$. The spring is driven by a force $F(t) = \sin(t/\mathrm{s})\,\mathrm{kg\,m/s^2}$, where t is the time. When $t = 0$, the spring is at rest in its neutral position. Find a formula describing the position of the spring as a function of time.

   We let $y(t)$ be the position at time t. Then $y(t)$ satisfies the following initial value problem:

$$(1\,\mathrm{kg})y'' + (4\,\mathrm{kg/s})y' + (3\,\mathrm{kg/s^2})y = \sin(t/\mathrm{s})\,\mathrm{kg\,m/s^2}, \quad \mathrm{y}(0) = 0\,\mathrm{m}, \quad \mathrm{y}'(0) = 0\,\mathrm{m/s}$$

   We begin by solving the homogeneous equation $(1\,\mathrm{kg})y'' + (4\,\mathrm{kg/s})y' + (3\,\mathrm{kg/s^2})y = 0$. The auxiliary equation is $(1\,\mathrm{kg})r^2 + (4\,\mathrm{kg/s})r + (3\,\mathrm{kg/s^2}) = 0$, with solutions $r = -1/\mathrm{s}$ and $r = -3/\mathrm{s}$. Thus, the homogeneous solutions are:

$$y_c = C_1 e^{-t/\mathrm{s}} + C_2 e^{-3t/\mathrm{s}}$$

   We now use the method of undetermined coefficients to find a particular solution. Since the driving force is proportional to $\sin(t/\mathrm{s})$, we guess a particular solution of the form

$$y_p = A\cos(t/\mathrm{s}) + B\sin(t/\mathrm{s})$$

   Then $y_p' = -(A/\mathrm{s})\sin(t/\mathrm{s}) + (B/\mathrm{s})\cos(t/\mathrm{s})$, and $y'' = -(A/\mathrm{s^2})\cos(t/\mathrm{s}) - (B/\mathrm{s^2})\sin(t/\mathrm{s})$. Thus, the left-hand-side of the equation we are trying to solve is:

$$\begin{aligned}
&(1\,\mathrm{kg})\left(-(A/\mathrm{s^2})\cos(t/\mathrm{s}) - (B/\mathrm{s^2})\sin(t/\mathrm{s})\right) + \\
&+ (4\,\mathrm{kg/s})\left(-(A/\mathrm{s})\sin(t/\mathrm{s}) + (B/\mathrm{s})\cos(t/\mathrm{s})\right) + \\
&\qquad + (3\,\mathrm{kg/s^2})\left(A\cos(t/\mathrm{s}) + B\sin(t/\mathrm{s})\right) = (\mathrm{kg/s^2})\left(-A + 4B + 3A\right)\cos(t/\mathrm{s}) + \\
&\qquad\qquad + (\mathrm{kg/s^2})\left(-B - 4A + 3B\right)\sin(t/\mathrm{s})
\end{aligned}$$

   where on the right-hand-side we have combined like terms in sin and cos. This should equal $(\mathrm{kg\,m/s^2})\sin(t/\mathrm{s})$, and so we see that:

$$2A + 4B = 0 \qquad\qquad -4A + 2B = 1\,\mathrm{m}$$

   We can solve this system of equations. The answer is:

$$A = -\frac{1}{5}\,\mathrm{m} \qquad\qquad B = \frac{1}{10}\,\mathrm{m}$$

   Therefore, we have:

$$\begin{aligned}
y_p &= -\frac{1}{5}\,\mathrm{m}\cos(t/\mathrm{s}) + \frac{1}{10}\,\mathrm{m}\sin(t/\mathrm{s}) \\
y_g &= y_p + y_c \\
&= C_1 e^{-t/\mathrm{s}} + C_2 e^{-3t/\mathrm{s}} + -\frac{1}{5}\,\mathrm{m}\cos(t/\mathrm{s}) + \frac{1}{10}\,\mathrm{m}\sin(t/\mathrm{s})
\end{aligned}$$

where y_g is the general solution.

Lastly, we solve the initial-value problem. We want $y(0) = 0$, and so

$$C_1 + C_2 - \frac{1}{5}\,\mathrm{m} = 0$$

We also want $y'(0) = 0$, and so:

$$-\mathrm{s}^{-1}C_1 - 3\mathrm{s}^{-1}C_2 + \frac{1}{10}\,\mathrm{m/s} = 0$$

We solve this system of equations:

$$C_1 = \frac{1}{4}\,\mathrm{m} \qquad C_2 = -\frac{1}{20}\,\mathrm{m}$$

Therefore the final answer is:

$$\boxed{y = \frac{1}{4}\mathrm{m}e^{-t/\mathrm{s}} - \frac{1}{20}\mathrm{m}e^{-3t/\mathrm{s}} - \frac{1}{5}\mathrm{m}\cos(t/\mathrm{s}) + \frac{1}{10}\mathrm{m}\sin(t/\mathrm{s})}$$