Math 1B: Quiz 9 ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (5 pts) Determine whether the following series converges or diverges. You do not need to find the limit. You must specify which convergence test(s) you are using, and why the conditions for the test are satisfied.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

The function $f(x) = \dfrac{1}{x\sqrt{\ln x}} = \dfrac{1}{x}\sqrt{\dfrac{1}{\ln x}}$ is a product of positive decreasing functions (the square root of a positive decreasing function is positive and decreasing), and so $f(x)$ is a positive decreasing function. Thus, the sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges if and only if the integral $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ converges, by the integral test. We evaluate this integral by substitution $u = \ln x$, whence $du = \frac{1}{x}dx$, and $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}du = \int_{\ln 2}^{\infty} \frac{du}{u^{1/2}}$. This integral diverges by the P-test with $p = 1/2 \leq 1$, and so the original sum also $\boxed{\text{diverges}}$.

2. (5 pts) Determine whether the following series converges or diverges. You do not need to find the limit. You must specify which convergence test(s) you are using, and why the conditions for the test are satisfied.

$$\sum_{n=1}^{\infty} \frac{2n-1}{n^3 + \arctan n}$$

We let $a_n = \dfrac{2n-1}{n^3 + \arctan n}$ and $b_n = \dfrac{2}{n^2}$; we are led to this choice of b_n by considering only the highest terms in the numerator and denominator of a_n. Then $a_n, b_n \geq 0$ for each n, and it's clear that $2n-1 \leq 2n$ and $n^3 + \arctan n \geq n^3$, so $a_n \leq b_n$. Moreover, the sum $\sum_1^{\infty} b_n$ converges by the P-test with $p = 2 > 1$, and so $\sum a_n$ $\boxed{\text{converges}}$ by the comparison test. If we instead use the limit comparison test, we see that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n-1)/(2n)}{(n^3 + \arctan n)/n^3} = \frac{1}{1} = 1$, since $\arctan n \leq \pi/2$, so $\lim_{n\to\infty} (\arctan n)/n^3 \leq \lim_{n\to\infty} (\pi/2)/n^3 = 0$. Again, the limit comparison test says that $\sum a_n$ converges if $\sum b_n$ does, which it does by the P-test.