Math 1B: Quiz 11 ANSWERS

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) Find a power series representation (centered at 0) of the function $f(x) = \ln(1 + \frac{x}{2})$. Your final answer must be in the form $\sum c_n x^n$. Find the interval of convergence of your series.

The easy way to do this quiz is to have some prior knowledge about power series for ln: $\ln(1+x) = \sum_1^\infty (-1)^{n-1} x^n/n$, so $\ln(1+\frac{x}{2}) = \sum_1^\infty \frac{(-1)^n}{2^n n} x^n$. We'll do it the careful way:

$$
\begin{aligned}
\frac{d}{dx}\left[\ln\left(1+\frac{x}{2}\right)\right] &= \frac{1/2}{1+\frac{x}{2}} \\
&= \frac{1}{2}\frac{1}{1-(-\frac{x}{2})} \\
&= \frac{1}{2}\sum_0^\infty \left(-\frac{x}{2}\right)^n \\
&= \frac{1}{2}\sum_0^\infty \left(-\frac{1}{2}\right)^n x^n \\
\ln\left(1+\frac{x}{2}\right) &= \int \frac{1}{2}\sum_0^\infty \left(-\frac{1}{2}\right)^n x^n dx \\
&= \frac{1}{2}\sum_0^\infty \left(-\frac{1}{2}\right)^n \frac{x^{n+1}}{n+1} + C \\
0 = \ln\left(1+\frac{0}{2}\right) &= \sum 0 + C = C \\
\ln\left(1+\frac{x}{2}\right) &= \frac{1}{2}\sum_0^\infty \left(-\frac{1}{2}\right)^n \frac{x^{n+1}}{n+1} \\
&= \sum_0^\infty \frac{(-1)^n}{2^{n+1}}\frac{x^{n+1}}{n+1} \\
&= \boxed{\sum_1^\infty \frac{(-1)^{n-1}}{2^n}\frac{x^n}{n}}
\end{aligned}
$$

We can now compute the interval of convergence. We use the ratio test, for example, to compute the radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1/(n2^n)}{1/((n+1)2^{n+1})}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{n}\right| = 2$$

At $x = 2$, the sum is alternating, and indeed is the Alternating Harmonic Series. It passes the Alternating Series Test since $\frac{1}{n}$ is a decreasing positive sequence with $\lim \frac{1}{n} = 0$. At $x = 2$, the sum is the (negative of the) Harmonic Series, and diverges by the integral test. Thus the interval of convergence is:

$$\boxed{x \in (-2, 2]}$$