Math 2135

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Announcements:

- HW1 will be posted by tomorrow morning at categorified.net/22Winter2135/

  It will be due by end of day Friday Jan 21.
- paid notetaker position available: notetaking@dal.ca
- supplementary videos: linear.axler.net/LADRvideos.html

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. These will be our "numbers" aka "scalars".

A vector space over $\mathbb{F}$ is a set V of "vectors" and two algebraic data:

* Given $v, w \in V$, $\underset{\in V}{v+w}$ should be defined
* Given $v \in V$ and $\lambda \in \mathbb{F}$, $\lambda \cdot v$ should be defined.

  $\lambda \cdot v \in V$

These data should satisfy:

* $(v+w)+u = v+(w+u)$] ← "$v+w+u$" makes sense.

$v+w = w+v$

$\exists$ "0" $\in V$ s.t. $v+0=v$ $\forall v$,

$\forall v \in V$, $\exists$ "$-v$" s.t. $v+(-v)=0$

mult of a scalar by a vector

* $(\lambda \cdot \mu) \cdot v = \lambda \cdot (\mu \cdot v)$] ← "$\lambda \mu v$" make sense.

$1 \cdot v = v$

addition in V

* $(\lambda+\mu) \cdot v = \lambda \cdot v + \mu \cdot v$

addition in $\mathbb{F}$

$\lambda \cdot (v+w) = \lambda \cdot v + \lambda \cdot w$

addition in V.

multiplication in $\mathbb{F}$.

<u>Remarks:</u>

1. were it not for $\exists 0$, $\emptyset = \{\}$ would be a v-space.
2. were it not for $1 \cdot v = v$, it would be valid to set $\lambda \cdot v = 0$ $\forall \lambda, v$.

If $V \neq \emptyset$ then "$\exists 0$" is overkill.

Lemma: $\forall v \in V$, $\underset{\underset{\mathbb{F}}{\uparrow}}{0} \cdot v = \underset{\underset{V}{\uparrow}}{0}$

Same argument shows: $\forall \lambda \in \mathbb{F}$, $\lambda \cdot 0 = 0$ ($\in V$).

Pf: $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$

subtract $0 \cdot v$ from both sides. $\square$

"$\exists -v$" is overkill:

Lemma: $\forall v \in V$, $(\underset{\underset{\mathbb{F}}{\uparrow}}{-1}) \cdot v = -v$

Pf: $0 = 0 \cdot v = (1 + (-1)) \cdot v = 1 \cdot v + (-1) \cdot v = v + (-1) \cdot v.$

Subtract v from both sides.

Defn: Let V be a v-space.

A subset $U \in V$ is a vector subspace aka sub-v-space

if U is also a v-space for the "same" $+$, $\circ$.

I.e: * If $v, u \in U \subseteq V$, so $v+u \in V$

I want $v+u \in U$.

* If $v \in U$ and $\lambda \in \mathbb{F}$, so $\lambda \cdot v \in V$

I want $\lambda \cdot v \in U$.

* I want $\boxed{0 \in U}$. equiv: $U \neq \emptyset$.

so no checking.

* all the other axioms are just equations that certainly hold for all elements of V, so they hold in U.

Examples: $(\mathbb{F} = \mathbb{R})$

- The set of all functions $f : \mathbb{R} \to \mathbb{R}$ is a v-space. "$\mathbb{R}^{\mathbb{R}}$".

$f, g : \mathbb{R} \to \mathbb{R}$,

$\underbrace{(f+g)(r) = f(r) + g(r)}$ $\forall r \in \mathbb{R}$

↑ defines the fn "$f+g$".

$\underbrace{(\lambda \cdot f)(r) = \lambda \cdot f(r)}$

↑ defines "$\lambda \cdot f$".

$\{\text{continuous fns}\} \subseteq \{\text{all functions}\}$.

It's a sub-v-space:

* If f, g both cont's, the $f+g$ is cont's. ✓
* If f cont's, then $\lambda \cdot f$ is cont's. ✓
* $0(r) := 0$. is a continuous fn. ✓

Fix real numbers a and b. $a=7$ $b=5$ no.

$a=7$ $b=0$ yes.

$\{$ functions $f:\mathbb{R}\to\mathbb{R}$ s.t. $f(\cancel{x})=\cancel{x}\}\subseteq\mathbb{R}^{\mathbb{R}}$.

x^2 $\cancel{x}$ 7 0

$f(x)=x-7$. $g(x)=x^2-49$

Is it a sub-v-space?

Yes if $a=$ anything, $b=0$.

No if $b\neq 0$.

<u>Check:</u> If f, g both satisfy $f(a)=b$, $g(a)=b$,

then $(f+g)(a)=b+b=2b \overset{?}{=} b$

First axiom holds iff $b=0$.

If f satisfies $f(a)=b$, then $\lambda f(a)=\lambda b \overset{?}{=} b$ $\forall\lambda$

holds iff $b=0$.

Does $0(a)=b$? Holds iff $b=0$.

Given v-subspaces $U_1, U_2, U_3 \subseteq V$

one fixed ambient v-space

<u>Define</u>: $U_1 + U_2 + U_3 \subseteq V$

to be the subset of V consisting of all vectors which can be constructed as $u_1 + u_2 + u_3$ for some $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$.

<u>E.g</u>: $U_1 \subseteq U_1 + U_2 + U_3$ because if $u_1 \in U_1$, then $u_1 + 0 + 0 = u_1$, and $0 \in U_2$ and $0 \in U_3$.

<u>Proposition</u>: $U = U_1 + U_2 + U_3$ is a v-subspace of V.

* $0 \in U$. ✓
* if $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$ with $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$

$v_1 \in U$ ...

Suppose $u, v \in U_1 + U_2 + U_3$.

i.e. $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$

for some $u_1, v_1 \in U_1$, $u_2, v_2 \in U_2$, $u_3, v_3 \in U_3$.

Then $u + v = \underbrace{(u_1 + v_1)}_{\in U_1} + \underbrace{(u_2 + v_2)}_{\in U_2} + \underbrace{(u_3 + v_3)}_{\in U_3}$ ✓

So $u + v = U_1 + U_2 + U_3$.

* If $u = u_1 + u_2 + u_3$, then $\lambda u = \underbrace{\lambda u_1}_{\in U_1} + \underbrace{\lambda u_2}_{\in U_2} + \underbrace{\lambda u_3}_{\in U_3}$ ✓.

Moreover: $U_1+U_2+U_3$ is the smallest v-subspace of V containing all of U_1, all of U_2, and all of U_3.

I.e: If $W \subseteq V$ is a v-subspace s.t. $U_1, U_2, U_3 \subseteq W$ then $U_1+U_2+U_3 \subseteq W$.

If $h: \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} h(x) & \text{if } x \neq 7 \\ 0 & \text{if } x = 7. \end{cases}$$

$$g(x) := \begin{cases} 0 & \text{if } x \neq 7 \\ h(7) & \text{if } x = 7. \end{cases}$$

E.g: $U_1 = \{ f: \mathbb{R} \to \mathbb{R} \text{ s.t. } f(7)=0 \} \subseteq \mathbb{R}^{\mathbb{R}}$ is v-subspace.

$U_2 = \{ g: \mathbb{R} \to \mathbb{R} \text{ s.t. } g(5)=0 \} \subseteq \mathbb{R}^{\mathbb{R}}$ is v-subspace

$U_1 + U_2 = \mathbb{R}^{\mathbb{R}}$.

any fn $\mathbb{R} \to \mathbb{R}$ can be expressed as $f+g$ where $f(7)=0$ and $g(5)=0$. Lots of ways!!

The sum $U_1 + U_2 + U_3$ is <u>direct</u> if there's no redundant ways of writing elements as $u_1 + u_2 + u_3$.

i.e. The sum is direct when, if

$$v = \underset{\underset{U_1}{\uparrow}}{u_1} + \underset{\underset{U_2}{\uparrow}}{u_2} + \underset{\underset{U_3}{\uparrow}}{u_3} = \underset{\underset{U_1}{\uparrow}}{u_1'} + \underset{\underset{U_2}{\uparrow}}{u_2'} + \underset{\underset{U_3}{\uparrow}}{u_3'}$$

then $u_1 = u_1'$ and $u_2 = u_2'$ and $u_3 = u_3'$.

<u>Why?</u> If any v had two ways of being written, then $v - v$ would as well.

<u>Remark</u>: It's enough to check whether 0 has an expression as $u_1 + u_2 + u_3$ other than $0+0+0$.

If X, Y are sets,

then (in math) $Y^X := \{$fns from X to $Y\}$

a typical element of this set

is called $f : X \to Y$.

In computer science, "$X \to Y$"

means the set that I call Y^X.

If a mathematician just writes "$X \to Y$", they probably mean $f : X \to Y$ but didn't want to choose a letter.