Math 2135

- HW 1 due Friday
  Typo in original version has been corrected.
- In person classes begin Monday Jan 31,
  i.e. two weeks from today, in LSC Common Area C244.
- OH this week T Th 1-2:30pm.
  Zoom link on course website: 955 8804 6787
  pwd: Chase-214

Let V is a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Goal: measure "size" of V. motivation: $\mathbb{R}^3$ ">" $\mathbb{R}^2$.

sizes: 3 2.

Given a vector $v \in V$, consider the set

$$\mathbb{F} \cdot v \subseteq V$$

of all vectors of the form λv for some $\lambda \in \mathbb{F}$.

<u>Lemma</u>: $\mathbb{F}v$ is a vector subspace.

<u>Pf</u>: To prove this, we need to show:

(a) $0 \in \mathbb{F}v$.

(b) If $u, w \in \mathbb{F}v$ then $u + w \in \mathbb{F}v$.

(c) If $u \in \mathbb{F}v$ and $\alpha \in \mathbb{F}$, then $\alpha u \in \mathbb{F}v$.

<u>Pf of (a)</u>: We know that for any v,

$$0 = 0 \cdot v.$$

where LHS is the 0 vector, and on the RHS 0 means 0 scalar, and so 0 has an expression as $\lambda \cdot v$ for $\lambda = 0 \in \mathbb{F}$.

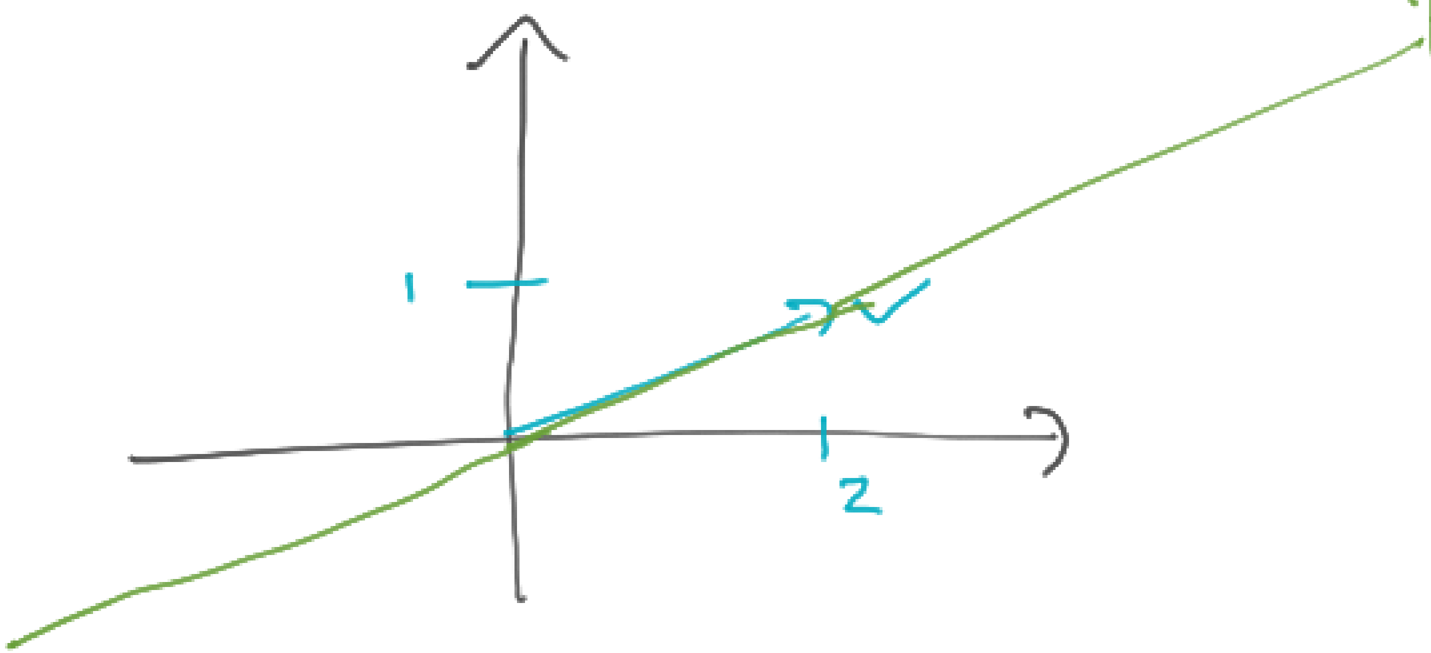
Pf of (c): $\alpha \cdot \lambda v = \alpha\lambda v \in \mathbb{F}v.$

Since $u \in \mathbb{F}v$, there is some $\lambda \in \mathbb{F}$ s.t. $u = \lambda v$, and so $\alpha u = \alpha\lambda v$. But $\alpha\lambda \in \mathbb{F}$, so $\alpha\lambda v \in \mathbb{F}v$.

Pf of (b): $\alpha v + \beta v = (\alpha+\beta)v.$ $\square$.

Example: In $\mathbb{R}^2$, $v = (2,1)$

$$\mathbb{F}v = \{\alpha \cdot (2,1) \text{ s.t. } \alpha \in \mathbb{F}\}$$
$$= \{(2\alpha, \alpha) \text{ s.t. } \alpha \in \mathbb{F}\}.$$



Example: If $v = 0$, (V arbitrary), then $\mathbb{F}v = \{0\}$.

Given V, we now have a bunch of vector subspaces namely the $\mathbb{F}v$'s w/ $v \in V$ varying,

summing vector subspaces produces more.

Take $v_1, \ldots, v_m$ some list of vectors.

Then $\mathbb{F}v_1, \mathbb{F}v_2, \ldots, \mathbb{F}v_m$ is a list of subspaces.

Can form their sum $\mathbb{F}v_1 + \mathbb{F}v_2 + \ldots + \mathbb{F}v_m$.

$\mathbb{F}v_1 + \dots + \mathbb{F}v_m$ is the set of vectors

which can be written as $u_1 + u_2 + \dots + u_m$

where $u_k \in \mathbb{F}v_k$ for all k.

i.e. where $u_k = \lambda_k v_k$ for some λ_k's.

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this set is the set of vectors of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m$$

where $\lambda_1, \dots, \lambda_m$ is any list of scalars.

<u>Defn</u>: A vector of this form is called a <u>linear combo</u> of $v_1, \dots, v_m$.

$$\mathbb{F}v_1 + \dots + \mathbb{F}v_m =: \operatorname{span}(v_1, \dots, v_m).$$

<u>E.g.</u>: $\operatorname{Span}((1,1),(1,2),(1,3)) \subseteq \mathbb{R}^2$

$=$ all vectors v s.t. $\exists\ \alpha, \beta, \gamma$ s.t.

$$v = \alpha(1,1) + \beta(1,2) + \gamma(1,3)$$

$$= (\alpha+\beta+\gamma,\ \alpha+2\beta+3\gamma).$$

Any v can be written this way.

i.e. $\forall (x,y)$, $\exists$ soln (α, β, γ) for the system of equations

$$x = \alpha + \beta + \gamma$$
$$y = \alpha + 2\beta + 3\gamma.$$

If $\operatorname{span}(v_1, \ldots, v_m) = V$

then say that $v_1, \ldots, v_m$ is a <u>spanning set</u>.

Or equiv: V is <u>spanned</u> by $v_1, \ldots, v_m$.

<u>E.g.</u> $\mathbb{R}^2$ is spanned by $(1,1), (1,2), (1,3)$.

<u>Remark:</u> It is useful for bookkeeping to think of spanning sets, etc, as ordered lists.

But the span of a list of vectors only depends on the set of vectors in that list.

Idea: measure size(V) by size of a spanning set.

1st approximation to this idea:

Defn: V is finite dimensional if there exist a finite spanning set.

E.g. $\mathbb{F}^n$ is f.d.

because it is spanned by the coordinate vectors $e_1 = (1, 0, 0 \dots 0)$, $e_2 = (0, 1, 0, \dots 0)$, $\dots e_n = (0, 0, \dots, 0, 1)$

Indeed: $(a_1, a_2, \dots, a_n) = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$.

E.g.: $\mathbb{F}^0 = \{()\}$ is spanned by $\emptyset$.

empty set.

$(\ )$ surround lists. parentheses

$\{\ \}$ surround sets braces

$[\ ]$ used for other things. brackets

E.g. (a) is the list of length 1 whose only entry is a.

$(\ )$ is the list of length 0.

$\emptyset = \{\ \}$ is $\underbrace{\text{the set with nothing in it.}}_{\text{aka the empty set.}}$

In any vector space V,

$\text{Span}(\underbrace{\emptyset}_{\uparrow \text{ set of size zero}})$ = all vectors which can be written as a sum of zero things and those things are multiples of elements of $\emptyset$.

The vector $0 \in V$ is a sum of zero things.

what is the product of zero things?