Math 2135

$\mathbb{F}$ = either $\mathbb{R}$ or $\mathbb{C}$

V = some v-space over $\mathbb{F}$.

Reminders:

1. HW1 due Friday
2. This week OH Th 1-2:30 instead of W 1-2:30.
3. In person instruction begins Jan 31.

Last time: a set or list of vectors $v_1, \dots, v_m \in V$ spans V if any vector $v \in V$ can be written as a linear combo of the v_k's, i.e. $\exists\, \alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t.

$$v = \alpha_1 v_1 + \dots + \alpha_m v_m.$$

E.g. in $\mathbb{R}^3$, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad v_k = \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}$

$a = \alpha_1 a_1 + \dots + \alpha_m a_m$

$b = \alpha_1 b_1 + \dots + \alpha_m b_m$

$c = \alpha_1 c_1 + \dots + \alpha_m c_m$

Rough idea: Spanning lists are long enough that this system of eqns has more unknowns than eqns

If $\exists$ finite spanning set then V is called finite dimensional. Otherwise V is infinite dimensional.

Example of a vector space which is infinite dim:

A polynomial is a function $f: \mathbb{F} \to \mathbb{F}$ s.t.

$\exists\, m \in \mathbb{N}$ and list $a_0, \dots, a_m \in \mathbb{F}$

$\leftarrow$ a list of length $m+1$

s.t. $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_m x^m$.

$\mathbb{F}[x]$ = set of all polynomials.

Called the coefficients of f.

__Calculus fact:__ If f is a polynomial, then its coefficients are uniquely determined. In other words, there cannot be more than one formula for f.

__Pf:__ If f is a poly, then $a_k = \frac{1}{k!} \left. \frac{d^k f}{dx^k} \right|_{x=0}$

$\Box$.

__i.e:__ If

$$f(x) = a_0 + a_1 x + \dots + a_m x^m$$
$$= b_0 + b_1 x + \dots$$

then $a_0 = b_0$ and $a_1 = b_1$ and ...

Observe: $\mathbb{F}[x]$ is a subspace of $\mathbb{F}^{\mathbb{F}}$.

i.e. (a) sum of polys is a poly.

(b) poly × number = poly.

(c) 0 is a poly.

$\uparrow$ all coefs $= 0$.

So $\mathbb{F}[x]$ is itself a vector space

Claim: $\mathbb{F}[x]$ is infinite dimensional.

Pf: Defn: a poly $f(x)$ has deg $\leq m$ if it can be written as $a_0 + \dots + a_m x^m$ for that m.

Remark: $\leq$ rather than $=$ is because maybe $a_m = 0$.

So for each poly f, $\exists$ some m s.t.
$\deg f \leq m$. ↑ depending on f.

Let $f_1(x), \dots, f_n(x)$ be any finite set / list of polynomials. Each one has deg $\leq$ some finite #. So there is a largest of these finite bounds, call it N.

i.e. all $f_1, \dots, f_n$ have deg $\leq N$.

But then any linear combo of them also has deg $\leq N$. So x^{N+1} is not in their span.

So this list doesn't span $\mathbb{F}[x]$.
But this list was arbitrary.
So $\mathbb{F}[x]$ does not admit a finite spanning set. So it's infinite dim'l.

$\mathbb{R}^2$ is also spanned by v_1, v_3.

Example: $\mathbb{R}^2$ is spanned by the set
$v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 (1,1)$.

But this spanning set is bigger than necessary,

next goal is to make this idea sharp.

Because $\mathbb{R}^2$ is also spanned by
$(1,0)$, $(0,1)$.

$\mathbb{R}^2$ is spanned by $v_1 = (1,0)$, $v_3 = (1,1)$.

Is this set bigger than necessary?

<u>No:</u> $\mathbb{R}^2$ is not spanned by v_1 alone

and it is not spanned by v_3 alone.

i.e. it's not true that every vector in $\mathbb{R}^2$

can be written as $\alpha_3 v_3$ for $v_3 = (1,1)$

and α_3 arbitrary scalar.

$(1,0)$, $(1,1)$, $(2,2)$

either of these is droppable, but not both at the same time.

not droppable

Defn: A list of vectors $v_1, v_2, \dots, v_m$ is linearly independant

if the only way of writing 0 as a linear combo of the v_k's

is $0 = 0v_1 + 0v_2 + \dots + 0v_m$.

i.e a choice of #s $\alpha_1 \dots \alpha_m \in \mathbb{F}$ s.t. $0 = \alpha_1 v_1 + \dots + \alpha_m v_m$

If there are other ways of writing 0 as lin combo, then $v_1, \dots, v_m$ is called linearly dependent.

① If $v_1, \dots, v_m$ is linearly indep,

then for any v, if v can be written as a lin combo of $v_1, \dots, v_m$, then this can be done in only one way.

<u>Indeed</u>: Suppose

$$v = \alpha_1 v_1 + \dots + \alpha_m v_m = \beta_1 v_1 + \dots + \beta_m v_m.$$

Then $0 = v - v = (\alpha_1 - \beta_1) v_1 + \dots + (\alpha_m - \beta_m) v_m$.

So $\alpha_1 - \beta_1 = 0$ and $\dots$ and $\alpha_m - \beta_m = 0$.

Examples

* $\emptyset$ is linearly indep.

Indeed, the only way of writing 0 as a sum of multiples of elts of $\emptyset$ is $0 =$ this space intentionally left blank...

* $\{0\}$ is linearly dependent.

because $0 = 1 \cdot 0$ is a way of writing 0 as a lin combo of 0.

* if $v \neq 0$, then $\{v\}$ is lin indep.

If $v_1, \ldots, v_m$ is linear dependant,

then pick some $\boxed{\text{nontrivial}}$ ← not all α's are 0.

$$0 = \alpha_1 v_1 + \ldots + \alpha_m v_m$$

So at least one $\alpha_k \neq 0$.

$-\alpha_k v_k =$ combo of the rest of them

$$= \alpha_1 v_1 + \ldots + \widehat{\alpha_k v_k} + \ldots + \alpha_m v_m$$

^ means "remove this one".

$$v_k = \boxed{-\frac{\alpha_1}{\alpha_k}} v_1 + \ldots \underbrace{}_{v_k \text{ is removed}} + \ldots + \boxed{-\frac{\alpha_m}{\alpha_k}} v_m.$$

in $\mathbb{F}$ because $\alpha_k \neq 0$.

$v_1, \ldots, v_m$ is lin dep
exactly when $\exists$ entry on the list which is in the span of the rest of the list.

<u>Defn:</u> A <u>basis</u> for V is a lin indep spanning set.

<u>Goal:</u> If V is f.dim, then bases exist and they all have the same size.
Once we've done this, we'll define $\dim(V) =$ size of any basis.