Math 2135: Linear Algebra

Final exam - practice

Your name:

University academic honour statement:

Dalhousie University has adopted the following statement, based on “The Fundamental Values of Academic Integrity” developed by the International Center for Academic Integrity (ICAI):

> Academic integrity is a commitment to the values of learning in an academic environment. These values include honesty, trust, fairness, responsibility, and respect. All members of the Dalhousie community must acknowledge that academic integrity is fundamental to the value and credibility of academic work and inquiry. We must seek to uphold academic integrity through our actions and behaviours in all our learning environments, our research, and our service.

Please **sign this page** to confirm that you will uphold these values, and that the work you submit on this exam is your own.

Exam structure

Part A contains six short unrelated questions, worth five points each.
Part B contains one longer question worth ten points.

Part A.

This section presents you with six statements. **Every one of them is false.** Give a short (one or two sentence) explanation of why.

1. **Suppose $U \subset V$ is a vector subspace, and $w_1, \ldots, w_m$ is a basis for the quotient space V/U. Choose elements $v_1, \ldots, v_m \in V$ such that $w_1 = v_1 + U$, ..., $w_m = v_m + U$. Then $v_1, \ldots, v_m$ spans V.**

   Since $\dim(V/U) = \dim(V) - \dim(U)$, in general $v_1, \ldots, v_m$ is too small to be a spanning set. The statement would be correct if "spans V" were replaced by "are linearly independent."

2. **If V is a vector space and $T \in \mathcal{L}(V)$ is injective, then T is invertible.**

   The statement is correct for finite-dimensional V, but fails for infinite-dimensional V.

3. **Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$ is a linear operator. Then the set of eigenvectors of T is a vector subspace of V.**

   For each λ, the set $E(\lambda, T)$ of (possibly-zero) eigenvectors with eigenvalue λ is a vector subspace. But the sum of (nonzero) eigenvectors with different eigenvalues is never an eigenvector.

4. **Suppose that V is a vector space, $T \in \mathcal{L}(V)$ is an operator, and $v_1, \ldots, v_m \in V$ are (nonzero) eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$. If $v_1, \ldots, v_m$ are linearly independent, then $\lambda_1, \ldots, \lambda_m$ are distinct.**

   The converse is true: if the $\lambda_1, \ldots, \lambda_m$ are distinct then $v_1, \ldots, v_m$ are linearly independent. But for example if T is the identity operator, then all vectors are eigenvectors with eigenvalue $\lambda = 1$.

5. **Suppose that V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $U_1 \subset V$ is a T-invariant vector subspace. Then there exists a T-invariant vector subspace $U_2 \subset V$ such that $V = U_1 \oplus U_2$ is a direct sum decomposition.**

   You can find a $U_2 \subset V$ such that $V = U_1 \oplus U_2$ is a direct sum decomposition, but there might not be any T-invariant ones. Examples can be built from non-diagonalizable operators. (And conversely the statement is true if T is diagonalizable.) For example, let $T \in \mathcal{L}(\mathbb{C}^2)$ be the operator $T(z, w) = (w, 0)$. Then the only one-dimensional invariant subspace is $U_1 = \ker(T) = \operatorname{im}(T) = \operatorname{span}((1, 0))$, and so any U_2 for which $U_1 \oplus U_2 = V$ cannot be invariant.

6. **There exists an operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that the only T-invariant subspaces of $\mathbb{R}^3$ are the trivial ones $\{0\}$ and $\mathbb{R}^3$.**

   Since 3 is odd, every operator $T \in \mathcal{L}(\mathbb{R}^3)$ admits an eigenvalue. Let v be a corresponding (nonzero) eigenvector. Then $\operatorname{span}(v)$ is a one-dimensional T-invariant subspace.

Part B.

Find all values of x such that the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & x \end{pmatrix}$$

is diagonalizable. Justify your answer.

Let us write $T \in \mathcal{L}(\mathbb{F}^3)$ for this operator. Since T is written in upper-triagonal form, its eigenvalues are its diagonal entries: 1, 5, and x. If these are distinct, then T has enough distinct eigenvalues and so is diagonalizable. In other words, all $x \neq 1, 5$ work. The only questions are $x = 1$ and $= 5$.

Suppose first that $x = 5$. Then the eigenvalues of T are 1 and 5. The eigenvectors are the kernels of the operators:

$$T - 1 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$
$$T - 5 = \begin{pmatrix} -4 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices both have rank 2. In other words, their images are 2-dimensional, and so their kernels are 1-dimensional. So the maximum size of a set of linearly independent eigenvectors is $1 + 1 = 2$. An eigenbasis, on the other hand, would have to have 3 vectors in it. So this matrix is not diagonalizable.

Now suppose that $x = 1$. Again the eigenvalues of T are 1 and 5, and the eigenvectors are the kernels of:

$$T - 1 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$
$$T - 5 = \begin{pmatrix} -4 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -4 \end{pmatrix}$$

$T - 1$ now has a 2-dimensional kernel, spanned by the vectors $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 3 \\ -2 \end{smallmatrix}\right)$. The kernel of $T - 5$ remains 1-dimensional, spanned by $\left(\begin{smallmatrix} 1 \\ -2 \\ 0 \end{smallmatrix}\right)$. These three vectors together form an eigenbasis, and so this T is diagonalizable.

In other words, the values of x such that T diagonalizable are $\boxed{\text{all numbers except } x = 5}$.