Math 2135: Linear Algebra

Assignment 1

Solutions

1. **Show that $\frac{1}{2}+\frac{\sqrt{3}}{2}i \in \mathbb{C}$ is a cube root of -1. Find two more cube roots of -1 in $\mathbb{C}$.**

   We compute:

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$= \left(\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)+\left(\frac{\sqrt{3}}{2}i\right)^2\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \left(\frac{1}{4}+\frac{\sqrt{3}}{2}i-\frac{3}{4}\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}\right) - + \left(\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}i\right)^2$$

$$= -\frac{1}{4}+0-\frac{3}{4} = -1$$

   Exactly the same calculations, with $-i$ in place of i, show that $\boxed{\frac{1}{2}-\frac{\sqrt{3}}{2}i}$ is a cube root of -1. And of course $\boxed{-1}$ is a (real) cube root of -1.

2. (a) **Does there exist a $\lambda \in \mathbb{C}$ such that $\lambda(2, 3i, 4+5i) = (6, 7i, 8-9i)$? Why or why not?**

   No. Note that the left-hand side is $(2\lambda, 3i\lambda, (4+5i)\lambda)$. For this to be equal to the right-hand side, we must have $2\lambda = 6$, which forces $\lambda = 3$. However, for this λ, we find that $3i\lambda = 9i \neq 7i$. So there is no equality.

   (b) **Does there exist a $\lambda \in \mathbb{C}$ such that $\lambda(2, 1+i) = (1-i, 1)$? Why or why not?**

   Yes. Indeed, $(1+i)(1-i) = 1^2 - i^2 = 1-(-1) = 2$. So $\lambda = 1/(1+i)$ works. What is this λ? It is

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2}-\frac{1}{2}i.$$

3. **Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Show that, if $\alpha \in \mathbb{F}$ and $v \in V$, and if $\alpha v = 0$, then either $\alpha = 0$ or $v = 0$ (or both).**

   Equivalently, we are asked to show that if $\alpha \neq 0$, then $v = 0$. But if $\alpha \neq 0$, then there is a number $\alpha^{-1} \in \mathbb{F}$ such that $\alpha\alpha^{-1} = 1$, in which case

$$0 = \alpha^{-1}0 = \alpha^{-1}\alpha v = 1v = v.$$

4. **The first three of the following six sets are subsets of the vector space $\mathbb{R}^3$, and the last three are subsets of the vector space $\mathbb{R}^{\mathbb{R}}$. Three of these six subsets are**

actually vector subspaces, and the other three are not. For those that are, simply state that they are indeed vector spaces: you do not need to prove why. For those that are not, state that they are not vector spaces, and give a reason: explain one of the axioms for vector space that fails. (There might be more than one!)

(a) **The set of triples $(x, y, z) \in \mathbb{R}^3$ such that $x + y = z$.**

This $\boxed{\text{is}}$ a vector subspace of $\mathbb{R}^3$. Basically this is because the equation "$x + y = z$" is linear.

(b) **The set of triples $(x, y, z) \in \mathbb{R}^3$ such that $xy = z$.**

This $\boxed{\text{is not}}$ a vector subspace of $\mathbb{R}^3$. For example, the vector $v = (1, 1, 1)$ is in this subset, but $v + v = (2, 2, 2)$ is not, so it is not closed under $+$.

(c) **The set of triples $(x, y, z) \in \mathbb{R}^3$ such that x, y, z are nonnegative.**

This $\boxed{\text{is not}}$ a vector subspace of $\mathbb{R}^3$. For example, the vector $v = (1, 1, 1)$ is in this subset, but $-v = (-1, -1, -1)$ is not, so it is not closed under scalar multiplication. (It is closed under $+$, on the other hand.)

(d) **The set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$.**

This $\boxed{\text{is}}$ a vector subspace of $\mathbb{R}^{\mathbb{R}}$, basically because sums of products of smooth functions are again smooth. (A function is *smooth* when it has a continuous derivative, which has a continuous derivative, which has a continuous derivative, ad infinitum.)

(e) **The set of all functions $f : \mathbb{R} \to \mathbb{R}$ which solve the differential equation $f''(x) = f(x)$, where f'' denotes the second derivative of f.**

This $\boxed{\text{is}}$ a vector subspace of $\mathbb{R}^{\mathbb{R}}$, basically because the equation "$f'' = f$" is linear. Indeed, 0 solves $f'' = f$ (because the derivative of the zero function is again zero); if f_1, f_2 both solve $f'' = f$, then so does $f_1 + f_2$ (because derivative of a sum is sum of derivatives); and if $\alpha \in \mathbb{R}$ then $(\alpha f)' = \alpha f'$, and so multiplying a solution to $f'' = f$ by α produces another solution.

(f) **The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0)$ is an integer.**

This $\boxed{\text{is not}}$ a vector subspace of $\mathbb{R}^{\mathbb{R}}$. It does pass some tests: it contains the 0 function and is closed under $+$ and $-$. However, it is not closed under scalar multiplication. Indeed: the constant function $f(x) \equiv 1$ is in this set, but multiplying this by the number $\alpha = \frac{1}{2}$ produces the constant function $f(x) \equiv \frac{1}{2}$, which is not in this set.

5. **Is $\mathbb{R}$ naturally a vector space over $\mathbb{C}$? Is $\mathbb{C}$ naturally a vector space over $\mathbb{R}$?**

$V = \mathbb{C}$ $\boxed{\text{is}}$ naturally a vector space over $\mathbb{F} = \mathbb{R}$. First, $\mathbb{C}$ comes equipped with an addition law which satisfies all the requirements for addition in a vector space. To be a vector space over $\mathbb{R}$, the other datum $\mathbb{C}$ needs is a way of taking a "scalar" $\alpha \in \mathbb{R}$ and a "vector" $v \in \mathbb{C}$ and multiply them to produce a new "vector" $\alpha v \in \mathbb{C}$. Since the real numbers are naturally a subset of the complex numbers, we can use the complex multiplication for this: just read "αv" as that product of complex numbers. This multiplication law does indeed satisfy the necessary associativity and distributivity rules.

$V = \mathbb{R}$ $\boxed{\text{is not}}$ naturally a vector space over $\mathbb{F}\mathbb{C}$. It does come with an addition law satisfying all requirements. However, in order to be a vector space, we would need a way to multiply a "scalar" $\alpha \in \mathbb{C}$ and a "vector" $v \in \mathbb{R}$ and produce a "vector" $\alpha v \in \mathbb{R}$. Let's try to find such a multiplication law; by trying and failing, we will amass evidence that it might be impossible. (We will not produce a proof that it is impossible, however.) Well, we could remember that $\mathbb{R}$ is a subset of $\mathbb{C}$ and so evaluate αv as a complex multiplication. But the result will typically be a complex number which is not real. We could take just its real part. But that will not be suitably associative.

Actually proving that $V = \mathbb{R}$ is not naturally a vector space over $\mathbb{F} = \mathbb{C}$ is hard because it requires saying more carefully what the word "naturally" means. Let's agree that in order to be natural, the vector addition should be the addition in $\mathbb{R}$. Let's also agree that "naturality" requires that the multiplication law αv, where $\alpha \in \mathbb{C}$ and $v \in \mathbb{R}$, have the following property: if α happens to be real, then αv is means the usual multiplication of real numbers. With these agreements, we can prove that $\mathbb{R}$ is not naturally a vector space over $\mathbb{C}$. Indeed, suppose that it were, and take $\alpha = \sqrt{-1}$ and $v = 1$. Then, by supposition, there would be some real number $r = \alpha v$. But this is also equal to βv where $\beta = r \in \mathbb{R} \subset \mathbb{C}$ and still $v = 1$. So $(\alpha - \beta)v = 0$. By exercise 2, this forces $\alpha - \beta = 0$ or $v = 0$ or both. Well, $v \neq 0$, so it forces $\alpha - \beta = 0$. But $\alpha = \sqrt{-1}$ whereas β is real, and so we have a contradiction.

6. **Let $V := \mathbb{R}_{>0}$ denote the set of positive real numbers. Let's define, for $v, w \in V$, their "V-sum" to be $v +_V w := vw$, where the subscript on the left-hand side the subscript reminds that it is a new notion of "sum" in the set V, and on the right-hand side we mean the product of positive numbers. Also, for $\lambda \in \mathbb{R}$ and $v \in V$, let's define their "product" $\lambda \cdot_V v := v^\lambda$, where the right-hand side means the exponential of real numbers. Is this V, with these notions of addition and scalar multiplication, a vector space over $\mathbb{R}$? Explain.**

   This V, with these notions of addition and scalar multiplication, $\boxed{\text{is}}$ a vector space over $\mathbb{R}$!

   There is a sneaky reason for this. Given $v \in V$, consider the real number $r = \log v \in \mathbb{R}^1$; conversely, given $r \in \mathbb{R}^1$, consider the positive real number $v = \exp v \in V = \mathbb{R}_{>0}$. (Mathematicians typically write "exp" for the natural, i.e. base e, exponential function, and "log" for the natural, i.e. base e, logarithm. But any base will work so long as they are the same: if you want to think of log as the base-10 logarithm, then just decide that $\exp(r) := 10^r$.) These two functions $\exp : r \leftrightarrow v : \log$ provide a perfect matching between $\mathbb{R}^1$ and V. Moreover, they relate the usual addition and scalar multiplication in $\mathbb{R}^1$ to the funny versions for V.

   There is also a non-sneaky reason: we can just check the axioms directly. First, we need to check whether $+_V$ is commutative and associative: If $u, v, w \in V$, do we have $v +_V w \stackrel{?}{=} w +_V v$ and $(u +_V v) +_V w \stackrel{?}{=} u +_V (v +_V w)$? Yes, because these questions are equivalent to asking, for the usual multiplication of real numbers, whether $vw \stackrel{?}{=} wv$ and $(uv)w \stackrel{?}{=} u(vw)$, and the answers to both questions are yes. Second, we need to check whether there is an element $0_V \in V$ such that, for every $v \in V$, $0_V +_V v = v$? Yes: take $0_V := 1$, and use that for usual multiplication, $1v = v$. And is there negation? I.e. given $v \in V$, is there $-_V v$ such that $v +_V (-_V v) = 0_V$? Yes: take $-_V v := v^{-1}$. Third, we need to check some properties of multiplication. We need to check whether, given $\alpha, \beta \in \mathbb{R}$ and $v \in V$, is $\alpha \cdot_V (\beta \cdot_V v) \stackrel{?}{=} (\alpha\beta) \cdot_V v$? Yes, because this unpacks to the question of whether $(v^\alpha)^\beta \stackrel{?}{=} v^{\alpha\beta}$. We need to check whether, given $v \in V$, is $1 \cdot_V v \stackrel{?}{=} v$? Yes, because this unpacks to the question of whether $v^1 \stackrel{?}{=} v$. And last, we need to check some distributivity laws, which unpack to whether $(vw)^\alpha \stackrel{?}{=} v^\alpha w^\alpha$ and whether $v^{\alpha+\beta} \stackrel{?}{v}^\alpha v^\beta$.

7. **Let $V = \mathbb{R} \sqcup \{+\infty, -\infty\}$. In other words, we take the set of real numbers, and add two new elements to the set, named "$+\infty$" and "$-\infty$." Equip V with an**

"addition" law which is the usual addition in $\mathbb{R}$, extended by

$$v + (+\infty) = (+\infty) + v = \infty, \qquad v + (-\infty) = (-\infty) + v = -\infty,$$
$$(+\infty) + (+\infty) = (+\infty), \qquad (+\infty) + (-\infty) = (-\infty) + (+\infty) = 0, \qquad (-\infty) + (-\infty) = -\infty$$

Also, define a "scalar multiplication" law which is the usual multiplication in $\mathbb{R}$, and, if $\alpha \in \mathbb{R}$, then

$$\alpha(+\infty) = \begin{cases} -\infty, & \alpha < 0, \\ 0, & \alpha = 0, \\ +\infty, & \alpha > 0, \end{cases} \qquad \alpha(-\infty) = \begin{cases} +\infty, & \alpha < 0, \\ 0, & \alpha = 0, \\ -\infty, & \alpha > 0. \end{cases}$$

Is this V, with these notions of addition and scalar multiplication, a vector space over $\mathbb{R}$? Explain.

This V, with these notions of addition and scalar multiplication, $\boxed{\text{is not}}$ a vector space over $\mathbb{R}$. Almost every axiom fails. For example, associativity of $+$ fails: $((+\infty)+(+\infty))+(-\infty) = (+\infty) + (-\infty) = 0$, but $(+\infty) + ((+\infty) + (-\infty)) = (+\infty) + 0 = +\infty$.

8. **Let V be a vector space.**

   (a) **Prove that the intersection of two vector subspaces of V is always another vector subspace of V.**

   Let $U_1, U_2 \subset V$ be vector subspaces. To show that $U := U_1 \cap U_2$ is a vector subspace of V, it suffices to show: (i) $0 \in U$; (ii) if $u, v \in U$ then $u + v \in U$; (iii) if $u \in U$ and $\alpha \in \mathbb{R}$, then $\alpha u \in U$.

   To show (i), note that $0 \in U_1$ and $0 \in U_2$ and hence $0 \in U_1 \cap U_2 = U$.

   To show (ii), note that, if $u, v \in U$, then $u, v \in U_1$ and so $u + v \in U_1$, but also $u, v \in U_2$ so $u + v \in U_2$, and hence $u + v \in U_1 \cap U_2 = U$.

   To show (iii), note that, if $u \in U$, then $u \in U_1$, and so $\alpha u \in U_1$ for every $\alpha \in \mathbb{R}$. But also $u \in U_2$, and so $\alpha u \in U_2$. Since αu is in both U_1 and U_2, it is in $U = U_1 \cap U_2$.

   (b) **Prove that the union of two vector subspaces of V is another vector subspace of V only if one of the two original subspaces contains the other one.**

   Let $U_1, U_2 \subset V$ be vector subspaces. We are asked to show: the only way for $U_1 \cup U_2$ to be a vector subspace is if $U_1 \subset U_2$ or $U_2 \subset U_1$ (or both). Equivalently, we are asked to show that if $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$, then $U_1 \cup U_2$ is not a vector subspace.

   Well, if $U_1 \not\subset U_2$, then we can choose an element $u_1 \in U_1$ which is not also in U_2. Similarly, if $U_2 \not\subset U_1$, then we can choose an element $u_2 \in U_2$ which is not also in U_1. So let's make such choices. Then these u_1 and u_2 are both in $U_1 \cup U_2$. We will show, however, that their sum $u_1 + u_2$ is not in $U_1 \cup U_2$. This will prove that $U_1 \cup U_2$ is not closed under $+$, and so is not a vector subspace.

   First, we claim that $u_1 + u_2$ is not in U_1. If it were, then since also $u_1 \in U_1$ and since U_1 is a vector subspace, we would have $u_2 = (u_1 + u_2) - u_1 \in U_1$. But when we selected u_2, we explicitly selected it to not be in U_1. Second, we claim that $u_1 + u_2$ is not in U_2. If it were, then since also $u_2 \in U_2$ and since U_2 is a vector subspace, we would have $u_1 = (u_1 + u_2) - u_2 \in U_2$. But when we selected u_1, we explicitly selected it to not be in U_2. So both statements "$u_1 + u_2 \in U_1$" and "$u_1 + u_2 \in U_2$" are false. This means that $u_1 + u_2$ is also not in the union $U_1 \cup U_2$, which is what we wanted to show.

9. **Let V be a vector space.**

   (a) **Is the addition of vector subspaces associative? In other words, given three vector subspaces $U_1, U_2, U_3 \subset V$, is**

   $$U_1 + (U_2 + U_3) \stackrel{?}{=} (U_1 + U_2) + U_3$$

   Yes. Indeed, the left-hand side unpacks to the set of all vectors in V which can be written as $u_1 + (u_2 + u_3)$ for some elements $u_1 \in U_1$, $u_2 \in U_2$, and $u_3 \in U_3$. The right-hand side unpacks to the set of all vectors in V which can be written as $(u_1 + u_2) + u_3$ for some elements $u_1 \in U_1$, $u_2 \in U_2$, and $u_3 \in U_3$. The associativity of vector addition implies that these two sets are the same

   (b) **Is the addition of vector subspaces commutative? In other words, given two vector subspaces $U_1, U_2 \subset V$, is**

   $$U_1 + U_2 \stackrel{?}{=} U_2 + U_1$$

   Yes. After unpacking, the commutativity of vector addition implies that these two sets are the same.

   (c) **Is there a vector subspace "O" $\subset V$ such that $U + O = U$ for every vector subspace $U \subset V$?**

   Yes. The subspace $O := \{0\}$ works. Indeed, for this $U + O$ is the set of vectors of the form $u + o$ where $u \in U$ and $o \in O$. But the only choice for o is $o = 0$, and so $U + O$ is the set of vectors of the form $u + 0$ where $u \in U$. But this is precisely the set U.

10. **Let $U_1 \subset \mathbb{R}^{\mathbb{R}}$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if $x < 0$. Let $U_2 \subset \mathbb{R}^{\mathbb{R}}$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if $x \geq 0$.**

    (a) **Show that U_1 and U_2 are vector subspaces of $\mathbb{R}^{\mathbb{R}}$.**

    For each $U = U_1$ or U_2, it suffices to show: (i) $0 \in U$; (ii) if $f, g \in U$ then $f + g \in U$; (iii) if $f \in U$ and $\alpha \in \mathbb{R}$, then $\alpha f \in U$.

    Well, the zero function 0 certainly vanishes on all negative numbers, and so it is in U_1. It also vanishes on all nonnegative numbers, so it is in U_2. This confirms (i) for both U_1 and U_2.

    Suppose that $f, g \in U_1$, i.e. if $x < 0$, then $f(x) = g(x) = 0$. Then, continuing to assume that $x < 0$, we find that $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$. This shows that $f + g \in U_1$, establishing (ii) for U_1. Essentially the same proof works for U_2: suppose that $f, g \in U_2$, i.e. that if $x \geq 0$ then $f(x) = g(x) = 0$; thus if $x \geq 0$ then $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$; thus $f + g \in U_2$.

    Property (iii) is proved similarly. Suppose that $f \in U_1$ and that $\alpha \in \mathbb{R}$ and that $x < 0$. Then $(\alpha f)(x) = \alpha\, f(x) = \alpha\, 0 = 0$. Since this was true for all $x < 0$, we find that $\alpha f \in U_1$. Suppose that $f \in U_2$ and that $\alpha \in \mathbb{R}$ and that $x \geq 0$. Then $(\alpha f)(x) = \alpha\, f(x) = \alpha\, 0 = 0$. Since this was true for all $x \geq 0$, we find that $\alpha f \in U_1$

    (b) **Show that $U_1 + U_2 = \mathbb{R}^{\mathbb{R}}$.**

    We must show that, for any function $f : \mathbb{R} \to \mathbb{R}$, there exists functions $f_1 \in U_1$ and $f_2 \in U_2$ such that $f = f_1 + f_2$. Set

    $$f_1(x) := \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad f_2(x) := \begin{cases} 0, & x \geq 0, \\ f(x), & x < 0. \end{cases}$$

Then we find that indeed f_1 does satisfy the requirements to be in U_1 (i.e. it is a function which vanishes on negative numbers), and f_2 does satisfy the requirements to be in U_2 (i.e. it is a function which vanishes on nonnegative numbers). We can add these functions by considering the two cases:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = \begin{cases} f(x) + 0, & x \geq 0, \\ 0 + f(x), & x < 0. \end{cases}$$

But this is just $f(x)$ in all cases, and so $f_1 + f_2 = f$.

(c) **Show that the sum is direct.**

It suffices to show that $U_1 \cap U_2 = \{0\}$. In other words, we claim that the only function which is in both U_1 and U_2 is the zero function. Well, if $f \in U_1 \cap U_2$, then certainly $f \in U_1$ and so $f(x) = 0$ for all negative x. On the other hand, if $f \in U_1 \cap U_2$, then certainly $f \in U_2$ and so $f(x) = 0$ for all nonnegative x. But every real x is either negative or nonnegative, and so we see that if $f \in U_1 \cap U_2$, then $f(x) = 0$ for all x. And so $f = 0$, which is what we wanted to prove.