Math 2135: Linear Algebra

Assignment 1

due January 21, 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please indicate (by name) who you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. Show that $\frac{1}{2} + \frac{\sqrt{3}}{2} i \in \mathbb{C}$ is a cube root of -1. Find two more cube roots of -1 in $\mathbb{C}$.

2. (a) Does there exist a $\lambda \in \mathbb{C}$ such that $\lambda(2, 3i, 4+5i) = (6, 7i, 8-9i)$? Why or why not?
   (b) Does there exist a $\lambda \in \mathbb{C}$ such that $\lambda(2, 1+i) = (1-i, 1)$? Why or why not?

3. Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Show that, if $\alpha \in \mathbb{F}$ and $v \in V$, and if $\alpha v = 0$, then either $\alpha = 0$ or $v = 0$ (or both).

4. The first three of the following six sets are subsets of the vector space $\mathbb{R}^3$, and the last three are subsets of the vector space $\mathbb{R}^{\mathbb{R}}$. Three of these six subsets are actually vector subspaces, and the other three are not. For those that are, simply state that they are indeed vector spaces: you do not need to prove why. For those that are not, state that they are not vector spaces, and give a reason: explain one of the axioms for vector space that fails. (There might be more than one!)

   (a) The set of triples $(x, y, z) \in \mathbb{R}^3$ such that $x + y = z$.
   (b) The set of triples $(x, y, z) \in \mathbb{R}^3$ such that $xy = z$.
   (c) The set of triples $(x, y, z) \in \mathbb{R}^3$ such that x, y, z are nonnegative.
   (d) The set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$.
   (e) The set of all functions $f : \mathbb{R} \to \mathbb{R}$ which solve the differential equation $f''(x) = f(x)$, where f'' denotes the second derivative of f.
   (f) The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0)$ is an integer.

5. Is $\mathbb{R}$ naturally a vector space over $\mathbb{C}$? Is $\mathbb{C}$ naturally a vector space over $\mathbb{R}$? **Hint:** One of the answers is "yes" and the other is "no."

   Remark: The word "natural" in mathematics is very important, very powerful, and has multiple meanings, only some of which are mathematically sharp. In this question, "naturally" means "using the algebraic structures that are part of the definition of these sets." It turns out that you can make both answers "yes" by using perverse definitions of "sum" and "scalar multiplication." More precisely, it's possible to prove that there exist (infinitely many!) ways of defining these such that both answers are "yes," but these perverse operations are discontinuous, uncomputable, and have nothing to do with the usual addition and multiplication on the supposed vector space. Such operations are "unnatural."

6. Let $V := \mathbb{R}_{>0}$ denote the set of positive real numbers. Let's define, for $v, w \in V$, their "V-sum" to be $v +_V w := vw$, where the subscript on the left-hand side the subscript reminds that it is a new notion of "sum" in the set V, and on the right-hand side we mean the product of positive numbers. Also, for $\lambda \in \mathbb{R}$ and $v \in V$, let's define their "product" $\lambda \cdot_V v := v^\lambda$, where the right-hand side means the exponential of real numbers. Is this V, with these notions of addition and scalar multiplication, a vector space over $\mathbb{R}$? Explain.

7. Let $V = \mathbb{R} \sqcup \{+\infty, -\infty\}$. In other words, we take the set of real numbers, and add two new elements to the set, named "$+\infty$" and "$-\infty$." Equip V with an "addition" law which is the usual addition in $\mathbb{R}$, extended by

$$v + (+\infty) = (+\infty) + v = \infty, \qquad v + (-\infty) = (-\infty) + v = -\infty,$$
$$(+\infty) + (+\infty) = (+\infty), \qquad (+\infty) + (-\infty) = (-\infty) + (+\infty) = 0, \qquad (-\infty) + (-\infty) = -\infty$$

Also, define a "scalar multiplication" law which is the usual multiplication in $\mathbb{R}$, and, if $\alpha \in \mathbb{R}$, then

$$\alpha(+\infty) = \begin{cases} -\infty, & \alpha < 0, \\ 0, & \alpha = 0, \\ +\infty, & \alpha > 0, \end{cases} \qquad \alpha(-\infty) = \begin{cases} +\infty, & \alpha < 0, \\ 0, & \alpha = 0, \\ -\infty, & \alpha > 0. \end{cases}$$

Is this V, with these notions of addition and scalar multiplication, a vector space over $\mathbb{R}$? Explain.

8. Let V be a vector space.

   (a) Prove that the intersection of two vector subspaces of V is always another vector subspace of V.

   (b) Prove that the union of two vector subspaces of V is another vector subspace of V only if one of the two original subspaces contains the other one.

9. Let V be a vector space.

   (a) Is the addition of vector subspaces associative? In other words, given three vector subspaces $U_1, U_2, U_3 \subset V$, is

   $$U_1 + (U_2 + U_3) \stackrel{?}{=} (U_1 + U_2) + U_3$$

   (b) Is the addition of vector subspaces commutative? In other words, given two vector subspaces $U_1, U_2 \subset V$, is

   $$U_1 + U_2 \stackrel{?}{=} U_2 + U_1$$

   (c) Is there a vector subspace "O" $\subset V$ such that $U + O = U$ for every vector subspace $U \subset V$?

10. Let $U_1 \subset \mathbb{R}^{\mathbb{R}}$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if $x < 0$. Let $U_2 \subset \mathbb{R}^{\mathbb{R}}$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if $x \geq 0$.

    (a) Show that U_1 and U_2 are vector subspaces of $\mathbb{R}^{\mathbb{R}}$.

    (b) Show that $U_1 + U_2 = \mathbb{R}^{\mathbb{R}}$.

    (c) Show that the sum is direct.