Math 2135: Linear Algebra

Assignment 2

Solutions

1. (a) **Suppose that $\{v_1, v_2, v_3, v_4\}$ spans a vector space V. Show that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4\}$ also spans V.**

   Let $v \in V$ be a vector. Since $\{v_1, v_2, v_3, v_4\}$ spans, we can find numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$ such that

   $$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4.$$

   This can be rearranged to

   $$v = \alpha_1(v_1 + v_2) - \alpha_1 v_2 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \alpha_1(v_1 + v_2) + (\alpha_2 - \alpha_1)v_2 + \alpha_3 v_3 + \alpha_4 v_4.$$

   Repeating the trick gives:

   $$v = \alpha_1(v_1 + v_2) + (\alpha_2 - \alpha_1)(v_2 + v_3) + (\alpha_3 - \alpha_2 + \alpha_1)(v_3 + v_4) + (\alpha_4 - \alpha_3 + \alpha_2 - \alpha_1)v_4.$$

   Thus we have found numbers $\alpha_1', \alpha_2', \alpha_3', \alpha_4'$ such that $v = \alpha_1'(v_1+v_2)+\alpha_2'(v_2+v_3)+\alpha_3'(v_3+v_4)+\alpha_4' v_4$, namely $\alpha_1' = \alpha_1$, $\alpha_2' = \alpha_2-\alpha_1$, $\alpha_3' = \alpha_3-\alpha_2+\alpha_1$, and $\alpha_4' = \alpha_4-\alpha_3+\alpha_2-\alpha_1$.

   (b) **Suppose that $\{v_1, v_2, v_3, v_4\}$ is linearly independent. Show that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4\}$ is also linearly independent.**

   Suppose that we have found $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

   $$0 = \alpha_1(v_1 + v_2) + \alpha_2(v_2 + v_3) + \alpha_3(v_3 + v_4) + \alpha_4 v_4.$$

   We wish to show that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.
   Well, expanding and grouping like terms, we find:

   $$0 = \alpha_1 v_1 + (\alpha_1 + \alpha_2)v_2 + (\alpha_2 + \alpha_3)v_3 + (\alpha_3 + \alpha_4)v_4.$$

   Since $\{v_1, v_2, v_3, v_4\}$ is assumed linearly independent, we learn that $0 = \alpha_1 = \alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = \alpha_3 + \alpha_4$. But if $0 = \alpha_1$ and $0 = \alpha_1 + \alpha_2$, then $0 = \alpha_2$. Repeating the argument with the other equations gives $0 = \alpha_3 = \alpha_4$.

2. **For which numbers t is the set $\{(3, 1, 4), (2, -3, 5), (5, 9, t)\}$ a basis of $\mathbb{R}^3$?**

   This set is a basis provided it is spanning and linearly independent. Suppose that linear independence fails. Then we would have numbers $a, b, c \in \mathbb{R}$, not all zero, such that

   $$(0, 0, 0) = a(3, 1, 4) + b(2, -3, 5) + c(5, 9, t) = (3a + 2b + 5c, a - 3b + 9c, 4a + 5b + tc).$$

Comparing the first two terms and solving for c gives $c = -\frac{3}{5}a - \frac{2}{5}b$. Comparing the second terms and solving for c gives $c = -\frac{1}{9}a + \frac{1}{3}b$. Let's set these equal and rearrange the equation:

$$\begin{aligned}
-\frac{3}{5}a - \frac{2}{5}b &= -\frac{1}{9}a + \frac{1}{3}b \\
\left(-\frac{3}{5} + \frac{1}{9}\right)a &= \left(\frac{2}{5} + \frac{1}{3}\right)b \\
\left(-3 + \frac{5}{9}\right)a &= \left(2 + \frac{5}{3}\right)b \\
(-27+5)\,a &= (18+15)\,b \\
-22a &= 33b \\
-\frac{2}{3}ab
\end{aligned}$$

In particular, either both a and b are zero, in which case c is, or neither a nor b is. In other words, if we are to find a nontrivial linear dependence, then we'd better have $a \neq 0$, and so we might as well devide through by a and then redefine $b \rightsquigarrow \frac{b}{a}$ and $c \rightsquigarrow \frac{c}{a}$, and just decide that $a = 1$. Then $b = -\frac{2}{3}$, and looking again at $c = -\frac{3}{5}a - \frac{2}{5}b = -\frac{1}{9}a + \frac{1}{3}b$ gives $c = -\frac{1}{3}$.

This was just from looking at the first two terms. To have a linear dependence, we would also need

$$0 = 4a + 5b + tc = 4 - \frac{10}{3} - \frac{t}{3}$$

or equivalently $t = 2$.

We have shown: *The set* $\{(3,1,4),(2,-3,5),(5,9,t)\}$ *is linearly independent if and only if* $t \neq 2$.

We showed in class that a linearly independent set is a basis if and only if it has the correct size. This set does have the correct size to be a basis. *The set* $\{(3,1,4),(2,-3,5),(5,9,t)\}$ *is a basis if and only if* $t \neq 2$.

3. **Let U be the subspace of $\mathbb{C}^5$ defined by**

$$U := \{(z_1,z_2,z_3,z_4,z_5) \in \mathbb{C}^5 \textbf{ s.t. } 6z_1 = z_2 \textbf{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

(a) **Find a basis for U.**

Note that $\dim \mathbb{C}^5 = 5$ and so U is finite-dimensional. So we can build a basis from the ground up: start with the empty set $\emptyset$, and include vectors, while staying linearly independent the whole time, until we have to stop.

$\mathrm{Span}(\emptyset) = \{0\} \neq U$. So we include some nonzero vector. For example, $(1,6,0,0,0) \in U$.
$\mathrm{Span}((1,6,0,0,0))$ is not all of U. For example, $(0,0,2,-1,0) \in U \smallsetminus \mathrm{Span}((1,6,0,0,0))$.
$\mathrm{Span}((1,6,0,0,0),(0,0,2,-1,0))$ is not all of U. For example, $(0,0,3,0,-1) \in U \smallsetminus \mathrm{Span}((1,6,0,0,0),(0,0,2,-1,0))$.

The set $\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)\}$ is easily seen to be in U. Let's confirm that it is linearly independent. Well,

$$a(1,6,0,0,0) + b(0,0,2,-1,0) + c(0,0,3,0,-1) = (a,?,?,-b,-c)$$

and I didn't feel like working out the middle two entries. The only way for this sum to equal $(0,0,0,0,0)$ is if $a = b = c = 0$.

We claim that $\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)\}$ spans U. Suppose that $v = (z_1, z_2, z_3, z_4, z_5) \in U$. Consider the vector

$$v' = z_1(1,6,0,0,0) - z_4(0,0,2,-1,0) - z_5(0,0,3,0,-1) = (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5).$$

Then v' and v certainly agree in the first, fourth, and fifth entries. The second entry of $v' - v$ is $6z_1 - z_2$, which must be zero if $v \in U$. Similarly, the third entry is $z_2 + 2z_4 + 3z_5$, which also vanishes by definition of U.

So $v' = v$, and we have shown that $\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)\}$ spans. Since it was also linearly independent, it is a basis of U.

Remark: This is not the only solution.

(b) **Extend the basis in part (a) to a basis for $\mathbb{C}^5$.**

We start with our basis $\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)\}$ and continue to include vectors. Note that $(0,1,0,0,0) \notin U = \mathrm{Span}\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)\}$. So let's add it in. Another vector not in there: $(0,0,1,0,0)$.

We claim that $\{(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1),(0,1,0,0,0),(0,0,1,0,0)\}$ is a basis. Indeed:

$$a(1,6,0,0,0)+b(0,0,2,-1,0)+c(0,0,3,0,-1)+d(0,1,0,0,0)+e(0,0,1,0,0) = (a, 6a+d, 2b+3c+e, -b, -c$$

For this sum to equal $(0,0,0,0,0)$ we must have $a = b = c = 0$, in which case also $d = e = 0$. So it is linearly independent. Conversely, the sum does equal $(z_1, z_2, z_3, z_4, z_5)$ provided $a = z_1$, $b = -z_4$, $c = -z_5$, $d = z_2 - 6a = z_2 - 6z_1$, and $e = z_3 - 2b - 2c = z_3 + 2z_4 + 3z_5$.

Remark: This is not the only solution.

4. **We ended lecture on Monday having stated the following theorem, but we didn't supply a complete proof:**

   Theorem: Let V be a finite-dimensional vector space, and $U_1, U_2 \subset V$ two vector subspaces. Then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

This exercise asks you to work through the remainder of the proof. A complete proof of this theorem can be found in *Linear Algebra Done Right*, and you are welcome to read that discussion while thinking about this exercise. However, your answers must be in your own words. I recommend that you close the book before starting to write up your answers.

To review, at the end of lecture we said the following. Since $U_1 \cap U_2 \subset V$ is a vector subspace, and since V is finite-dimensional, we know that $U_1 \cap U_2$ is also finite-dimensional. Suppose that A is any basis for $U_1 \cap U_2$. Then we argued that we can extend A to a finite basis $A \cup B$ for U_1 and we can extend A to a finite basis $A \cup C$ for U_2. We asserted, but did not prove, that $A \cup B \cup C$ is a basis for $U_1 + U_2$.

Let's give names: $A = \{u_1, \ldots, u_k\}$, $B = \{v_1, \ldots, v_m\}$, and $C = \{w_1, \ldots, w_n\}$.

(a) **Implicit in the notation is that A and B are disjoint, and that A and C are disjoint. Explain why B and C are disjoint. (Two sets are *disjoint* if their intersection is empty, i.e. if there are no elements in common.)**

By definition, the elements of C are in U_2 but not U_1, whereas the elements of B are in U_1 (but not U_2).

(b) **Explain why the assertion "$A \cup B \cup C$ is a basis for $U_1 + U_2$" implies the theorem.**

The dimension of a subspace is the size of any of its bases. Since they are bases, $\#A = \dim(U_1 \cap U_2)$, whereas $\dim U_1 = \#(A \cup B) = \#A + \#B$ and $\dim U_2 = \$(A \cup C) = \#A + \#C$. Thus

$$\#(A \cup B \cup C) = \#A + \#B + \#C = (\#A + \#B) + (\#A + \#C) - \#A,$$

which is equal to the right-hand side of the equation we want to prove. On the other hand, if $A \cup B \cup C$ is a basis for $U_1 + U_2$, then also $\#(A \cup B \cup C)$ is equal to the left-hand side of the equation we want to prove.

(c) **Explain why $\mathrm{Span}(A \cup B \cup C) = U_1 + U_2$. In other words, explain why if**

$$v = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \ldots \gamma_1 w_k$$

where all the α's, β's, and γ's are in $\mathbb{F}$, then $v \in U_1 + U_2$, and conversely why any $v \in U_1 + U_2$ can be written as such a linear combination for some α's, β's, and γ's in $\mathbb{F}$.

If $v \in U_1 + U_2$, then we can find — we've run out of letters — let's say $x \in U_1$ and $y \in U_2$ such that $v = x + y$. Now, since $A \cup B$ is a basis for U_1, we can find numbers $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_m$ such that

$$x = \lambda_1 u_1 + \cdots + \lambda_k u_k + \mu_1 v_1 + \cdots + \mu_m v_m.$$

Similarly, since $A \cup C$ is a basis for U_2, we can find numbers $\rho_1, \ldots, \rho_k, \nu_1, \ldots, \nu_n$ such that

$$y = \rho_1 u_1 + \cdots + \rho_k v_k + \nu_1 w_1 + \cdots + \nu_n w_n.$$

Adding the two expressions gives

$$v = x + y = (\lambda_1 + \rho_1) u_1 + \cdots + (\lambda_k + \rho_k) u_k + \mu_1 v_1 + \cdots + \mu_m v_m + \nu_1 w_1 + \cdots + \nu_n w_n.$$

In other words, we can win by setting $\alpha_i = \lambda_i + \rho_i$, $\beta_i = \mu_i$, and $\gamma_i = \nu_i$.

(d) **(The most interesting part.) Explain why $A \cup B \cup C$ is linearly independent. In other words, we want to show that if**

$$0 = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \ldots \gamma_1 w_k$$

for some α's, β's, and γ's in $\mathbb{F}$, then all the α's, β's, and γ's are 0. So assume that you have found some such solution, and let

$$\begin{aligned} u &:= \alpha_1 u_1 + \cdots + \alpha_k u_k \\ v &:= \beta_1 v_1 + \cdots + \beta_m v_m \\ w &:= \gamma_1 w_1 + \ldots \gamma_1 w_k \end{aligned}$$

Explain why $v \in U_1$. Explain why also $v \in U_2$. Conclude that $v \in U_1 \cap U_2$. Explain why this implies that there are numbers $\delta_1, \ldots, \delta_k \in \mathbb{F}$ such that

$$v = \delta_1 u_1 + \cdots + \delta_k u_k.$$

Explain why this implies that either all the βs are zero or that $A \cup B$ is linearly dependent. (Consider the difference of two expressions for v.)
But $A \cup B$ is linearly independent by assumption (which assumption?), so all the βs are zero. Explain why this, together with the assumption (which one?) that $A \cup C$ is linearly independent, implies that all the αs and all the γs are zero.
With these u, v, w, we have

$$0 = u + v + w.$$

Note that $v \in \mathrm{Span}(B) \subset \mathrm{Span}(A \cup B) = U_1$. Note also that $u + w \in \mathrm{Span}(A \cup C) = U_2$. But $v = -(u + w)$, and so $v \in U_2$. So $v \in U_1 \cap U_2$.
But A is a basis for $U_1 \cap U_2$, so there must exist δ's as above. So

$$0 = v - v = (\delta_1 u_1 + \cdots + \delta_k u_k) - (\beta_1 v_1 + \cdots + \beta_m v_m) = \delta_1 u_1 + \cdots + \delta_k u_k + (-\beta_1) v_1 + \cdots + (-\beta_m) v_m.$$

If any of the δ's or β are nonzero, then we have found a linear dependency in $A \cup B$, which we assumed was a basis. So all the δs and all the βs are zero, and so $v = 0$.
But then our original assumed dependency becomes

$$0 = u + w = \alpha_1 u_1 + \cdots + \alpha_k u_k + \gamma_1 w_1 + \ldots \gamma_1 w_k.$$

On the other hand, since $A \cup C$ is linearly independent, it must happen that all the αs and all the γs vanish.

5. **Suppose that V is finite-dimensional and contains three vector subspaces U_1, U_2, U_3. The theorem in Exercise 4 might lead you to think that**

$$\begin{aligned}\dim(U_1 + U_2 + U_3) \stackrel{?}{=} \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) \\ - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3),\end{aligned}$$

but this formula is *not true* in general.

(a) **Explain why $V = \mathbb{R}^2$, with U_1, U_2, U_3 any three pairwise-distinct 1-dimensional subspaces, provides a counterexample.**
If $U_1, U_2 \subset \mathbb{R}^2$ are both one-dimensional, then either they are equal or they intersect only at the origin. In other words, if U_1, U_2, U_3 any three pairwise-distinct 1-dimensional subspaces, then we definitely have $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{0\}$, and all of these spaces have dimension 0. So the supposed equation becomes

$$\dim(U_1 + U_2 + U_3) \stackrel{?}{=} 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3.$$

But the sum $U_1 + U_2 + U_3$ is a subspace of $\mathbb{R}^2$, and so cannot have dimension > 2. So the supposed equation is false.
In fact, if U_1, U_2 are distinct one-dimensional subspaces in $\mathbb{R}^2$, then we already see from the theorem in Exercise 4 that $\dim(U_1 + U_2) = 1 + 1 - 0 = 2$, and so $U_1 + U_2 = \mathbb{R}^2$. Thus $U_1 + U_2 + U_3 = \mathbb{R}^2$ as well.

(b) **Suppose you tried to repeat the proof from Exercise 4 but with three subspaces. Which step of the proof fails? Explain.**

The step that fails is the "interesting" linear independence part. Let us explain:

Some subscripts will help manage the alphabet soup. Let's choose a basis A_{123} for $U_1 \cap U_2 \cap U_3$. We can extend it to a basis $A_{123} \cup A_{12}$ for $U_1 \cap U_2$, to a basis $A_{123} \cup A_{13}$ for $U_1 \cap U_3$, and to a basis $A_{123} \cup A_{23}$ for $U_2 \cap U_3$. Moreover, a version of the argument from Exercise 4 shows that $A_{123} \cup A_{12} \cup A_{13}$ is a linearly independent set in U, and so extends to a basis $A_{123} \cup A_{12} \cup A_{13} \cup A_1$ for U_1. By the same token, there are bases $A_{123} \cup A_{12} \cup A_{23} \cup A_2$ for U_2 and $A_{123} \cup A_{13} \cup A_{23} \cup A_3$ for U_3.

Then, to prove the supposed equation, it would suffice to prove that $A_{123} \cup A_{12} \cup A_{13} \cup A_{23} \cup A_1 \cup A_2 \cup A_3$ is a basis for $U_1 + U_2 + U_3$.

It is a spanning set, by a version of the argument from Exercise 4.

But when we try to prove linear independence, we run into trouble. For example, we might get into a situation where some vector v can be shown to be in U_1, and also in $U_2 + U_3$. If we could show it was in $U_1 \cup U_3$, then we could win: we'd be able to show it was zero through a version of the argument from Exercise 4. But all we can do is get into $U_2 + U_3$.

6. **Suppose, in Exercise 4, that V is infinite-dimensional. Does this really matter for the theorem? Explain. Hint: What happens if, even though V is infinite-dimensional, U_1 and U_2 are both finite-dimensional? What happens if one or both of them is infinite-dimensional?**

Suppose that U_1 and U_2 are both finite-dimensional. Then so is $U_1 \cap U_2$, and we can simply proceed with the argument: choose bases; show that something is a basis for something.

Suppose that $\dim U_1 = \infty$ but $\dim U_2 < \infty$. Then $\dim(U_1 + U_2)$ is also infinite, whereas $\dim(U_1 \cap U_2)$ is finite, and the equation we want to prove is the definitely-true statement

$$\infty = \infty + (\text{finite}) - (\text{finite}).$$

Ditto with the roles of U_1 and U_2 switched.

If U_1 and U_2 are both infinite-dimensional, then we have

$$\infty = \infty + \infty - \dim(U_1 \cap U_2).$$

If $U_1 \cap U_2$ is finite-dimenisonal, then this is definitely a true statement. If $\dim(U_1 \cap U_2) = \infty$, then probably we might simply throw our hands in the air and decide that the statement has no content.

Or we might decide: whatever these dimensions are, we definitely always have $\dim(U_2) \geq \dim(U_1 \cap U_2)$. So we should always interpret "$\dim(U_2) - \dim(U_1 \cap U_2)$" as being "positive," perhaps infinite, even if it is $\infty - \infty$ and so not a specific number. Then we could decide that, if $\dim U_1 = \infty$, then the right-hand side is $\infty + (\text{positive})$, which is undeniably infinite. With this interpretation, the theorem remains true in all cases.