Math 2135: Linear Algebra

Assignment 2

due 1 February 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please indicate (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. (a) Suppose that $\{v_1, v_2, v_3, v_4\}$ spans a vector space V. Show that $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\}$ also spans V.

   (b) Suppose that $\{v_1, v_2, v_3, v_4\}$ is linearly independent. Show that $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\}$ is also linearly independent.

2. For which numbers t is the set $\{(3, 1, 4), (2, -3, 5), (5, 9, t)\}$ a basis of $\mathbb{R}^3$?

3. Let U be the subspace of $\mathbb{C}^5$ defined by

$$U := \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \text{ s.t. } 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

   (a) Find a basis for U.

   (b) Extend the basis in part (a) to a basis for $\mathbb{C}^5$.

4. We ended lecture on Monday having stated the following theorem, but we didn't supply a complete proof:

   Theorem: Let V be a finite-dimensional vector space, and $U_1, U_2 \subset V$ two vector subspaces. Then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

   This exercise asks you to work through the remainder of the proof. A complete proof of this theorem can be found in *Linear Algebra Done Right*, and you are welcome to read that discussion while thinking about this exercise. However, your answers must be in your own words. I recommend that you close the book before starting to write up your answers.

   To review, at the end of lecture we said the following. Since $U_1 \cap U_2 \subset V$ is a vector subspace, and since V is finite-dimensional, we know that $U_1 \cap U_2$ is also finite-dimensional. Suppose that A is any basis for $U_1 \cap U_2$. Then we argued that we can extend A to a finite basis $A \cup B$ for U_1 and we can extend A to a finite basis $A \cup C$ for U_2. We asserted, but did not prove, that $A \cup B \cup C$ is a basis for $U_1 + U_2$.

   Let's give names: $A = \{u_1, \ldots, u_k\}$, $B = \{v_1, \ldots, v_m\}$, and $C = \{w_1, \ldots, w_n\}$.

(a) Implicit in the notation is that A and B are disjoint, and that A and C are disjoint. Explain why B and C are disjoint. (Two sets are *disjoint* if their intersection is empty, i.e. if there are no elements in common.)

(b) Explain why the assertion "$A \cup B \cup C$ is a basis for $U_1 + U_2$" implies the theorem.

(c) Explain why $\mathrm{Span}(A \cup B \cup C) = U_1 + U_2$. In other words, explain why if

$$v = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \ldots \gamma_1 w_k$$

where all the α's, β's, and γ's are in $\mathbb{F}$, then $v \in U_1 + U_2$, and conversely why any $v \in U_1 + U_2$ can be written as such a linear combination for some α's, β's, and γ's in $\mathbb{F}$.

(d) (The most interesting part.) Explain why $A \cup B \cup C$ is linearly independent. In other words, we want to show that if

$$0 = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \ldots \gamma_1 w_k$$

for some α's, β's, and γ's in $\mathbb{F}$, then all the α's, β's, and γ's are 0. So assume that you have found some such solution, and let

$$\begin{aligned} u &:= \alpha_1 u_1 + \cdots + \alpha_k u_k \\ v &:= \beta_1 v_1 + \cdots + \beta_m v_m \\ w &:= \gamma_1 w_1 + \ldots \gamma_1 w_k \end{aligned}$$

Explain why $v \in U_1$. Explain why also $v \in U_2$. Conclude that $v \in U_1 \cap U_2$. Explain why this implies that there are numbers $\delta_1, \ldots, \delta_k \in \mathbb{F}$ such that

$$v = \delta_1 u_1 + \cdots + \delta_k u_k.$$

Explain why this implies that either all the βs are zero or that $A \cup B$ is linearly dependent. (Consider the difference of two expressions for v.)

But $A \cup B$ is linearly independent by assumption (which assumption?), so all the βs are zero. Explain why this, together with the assumption (which one?) that $A \cup C$ is linearly independent, implies that all the αs and all the γs are zero.

5. Suppose that V is finite-dimensional and contains three vector subspaces U_1, U_2, U_3. The theorem in Exercise 4 might lead you to think that

$$\begin{aligned} \dim(U_1 + U_2 + U_3) \stackrel{?}{=} &\dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) \\ &- \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3), \end{aligned}$$

but this formula is *not true* in general.

(a) Explain why $V = \mathbb{R}^2$, with U_1, U_2, U_3 any three pairwise-distinct 1-dimensional subspaces, provides a counterexample.

(b) Suppose you tried to repeat the proof from Exercise 4 but with three subspaces. Which step of the proof fails? Explain.

6. Suppose, in Exercise 4, that V is infinite-dimensional. Does this really matter for the theorem? Explain. **Hint:** What happens if, even though V is infinite-dimensional, U_1 and U_2 are both finite-dimensional? What happens if one or both of them is infinite-dimensional?